Calibration of Stochastic Volatility Models: A Tikhonov Regularization Approach

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Outline

Local Volatility Model

$$\frac{dX_t}{X_t} = (r-q)dt + \sigma_{loc}(X_t, t)dW_t^0$$

- B-S equation remains valid

$$\frac{\partial V}{\partial t} + \frac{1}{2}x^2\sigma_{loc}^2(x,t)\frac{\partial^2 V}{\partial x^2} + (r-q)x\frac{\partial V}{\partial x} - rV = 0$$
$$V|_{t=T} = (x-K)^+ \qquad \text{in } x > 0, t < T$$

- How to identify $\sigma_{loc}(\cdot,\cdot)$ s.t.

$$V(x^*, t^*; K_i, T_j) = V^*(K_i, T_j),\ i,j = 1,2,...$$

- Calibration based on B-S Eq: Lagnado and Osher (1997)

Dupire Equation

- Dupire Equation (1994): as a function of K, T, V satisfies

$$\frac{\partial V}{\partial T} - \frac{1}{2}K^2\sigma_{loc}^2(K,T)\frac{\partial^2 V}{\partial K^2} + (r-q)K\frac{\partial V}{\partial K} + qV = 0$$
$$V|_{T=t^*} = (x^* - K)^+ \qquad \text{in} \;\; K > 0, T > t^*$$

- Calibration based on the Dupire equation: A standard inverse problem. See Bouchouev and Isakov (1997,1999), Jiang et al. (2001,2003), Egger and Engl (2005), etc.
- However, empirical evidences show that σ_t is not a deterministic function of X_t and t. See Avellaneda et al. (1996), Renault and Touzi (1996), Carmona and Xu (1997), Brockman and Chowdhury (1997), Dumas et al. (1998), Buraschi and Jackwerth (2001), Hagan et al. (2002), Fouque et al. (2003), etc.

Stochastic Volatility Model (SVM)

- Under the risk-neutral world,

$$\begin{aligned}\frac{dX_t}{X_t} &= (r-q)dt + \sigma_t dW_t^0 \\ dY_t &= b(Y_t,t)dt + \beta(Y_t,t)dW_t^1\end{aligned}$$

where $\sigma_t = f(Y_t)$, $dW_t^0 \cdot dW_t^1 = \rho dt$.

- The European call option price $V(x,y,t;K,T)$ satisfies

$$\begin{aligned}\mathcal{L}V &= 0 \text{ in } x>0, y>0, t<T \\ V|_{t=T} &= (x-K)^+\end{aligned}$$

where $\mathcal{L}V \equiv \frac{\partial V}{\partial t} + \frac{1}{2}x^2 f^2(y)\frac{\partial^2 V}{\partial x^2} + \rho\beta(y,t)xf(y)\frac{\partial^2 V}{\partial x \partial y} + \frac{1}{2}\beta^2(y,t)\frac{\partial^2 V}{\partial y^2}$
$$+(r-q)x\frac{\partial V}{\partial x} + b(y,t)\frac{\partial V}{\partial y} - rV$$

Calibration Problem

- Target: to identify $b = b(\cdot, \cdot)$ and $\beta = \beta(\cdot, \cdot)$ s.t. the model outputs match the market prices $V^*(K_i, T_j)$, $i, j = 1, 2, \cdots$.
- Historical work: $b(\cdot, \cdot)$ and $\beta(\cdot, \cdot)$ are assumed to have special structure, e.g.
  - The extended Hull-White (1987): $Y_t = \sigma_t^2$

$$dY_t = \kappa(\theta - Y_t)dt + \gamma Y_t dW_t^1$$

  - Heston (1993): $Y_t = \sigma_t^2$

$$dY_t = \kappa(\theta - Y_t)dt + \gamma \sqrt{Y_t} dW_t^1$$

  - Stein and Stein (1991): $Y_t = \sigma_t$

$$dY_t = \kappa(\theta - Y_t)dt + \gamma dW_t^1$$

  - Hagan et al. (2002): SABR model
- We aim to recover $b(\cdot, \cdot)$ and $\beta(\cdot, \cdot)$ directly.

Dupire Equation for SVM: Dai and Wu (2002)

- Differentiate the option pricing equation w.r.t. K twice

$$\mathcal{L}\left(\frac{\partial^2 V}{\partial K^2}\right) = 0 \quad \text{in } x > 0, y > 0, t < T$$
$$\frac{\partial^2 V}{\partial K^2}(x, y, T; K, T) = \delta(x - K)$$

- $\psi(x, y, t; K, \overline{y}, T)$ is the fundamental solution associate with $\mathcal{L}$

$$\frac{\partial^2 V}{\partial K^2}(x, y, t; K, T) = \int_0^\infty \psi(x, y, t; K, \overline{y}, T) d\overline{y}$$

- $\mathcal{L}^* \psi = 0, \quad K > 0, \overline{y} > 0, T > t^*$

where
$$\mathcal{L}^*\psi \equiv \frac{\partial \psi}{\partial T} - \frac{1}{2}\frac{\partial^2}{\partial K^2}(K^2 f^2(\overline{y})\psi) - \frac{\partial^2}{\partial K \partial \overline{y}}(\rho\beta(\overline{y})K f(\overline{y})\psi) - \frac{1}{2}\frac{\partial^2}{\partial \overline{y}^2}(\beta^2(\overline{y})\psi) + (r - q)\frac{\partial}{\partial K}(K\psi) + \frac{\partial}{\partial \overline{y}}(b(\overline{y}, T)\psi) + r\psi$$

Continued:

$$\frac{\partial V}{\partial T} + (r-q)K\frac{\partial V}{\partial K} + qV = \frac{1}{2}K^2\int_0^\infty f^2(\overline{y})\psi(x^*,y^*,t^*;K,\overline{y},T)d\overline{y}$$

$$V|_{T=t^*} = (x^*-K)^+ \qquad \text{in } K>0, T>t^*$$

whose solution is

$$V(x^*,y^*,t^*;K,T) = e^{-q(T-t^*)}\left(x^* - e^{-(r-q)(T-t^*)}K\right)^+$$

$$+\int_{t^*}^T\int_0^\infty \frac{K^2}{2}e^{(-2r+q)(T-\tau)}f^2(\overline{y})\psi(x^*,y^*,t^*;Ke^{-(r-q)(T-\tau)},\overline{y},\tau)d\overline{y}d\tau$$

Here $\psi(x^*,y^*,t^*;K,\overline{y},T)$, as a function of K, $\overline{y}$, and T, satisfies

$$\begin{aligned} \mathcal{L}^*\psi &= 0 \quad \text{in } K>0, \overline{y}>0, T>t^* \\ \psi|_{T=t^*} &= \delta(x^*-K)\delta(y^*-\overline{y}) \end{aligned}$$

Derivation: Probabilistic Approach

- Dupire (1994)

$$\sigma^2_{loc}(K,T) = \frac{\frac{\partial V}{\partial T} + (r-q)K\frac{\partial V}{\partial K} + qV}{\frac{1}{2}K^2\frac{\partial^2 V}{\partial K^2}}$$

- Derman and Kani (1998)

$$\sigma^2_{loc}(K,T) = E[\sigma^2_T | X_T = K]$$

We then infer

$$\begin{aligned}
\frac{\partial V}{\partial T} + (r-q)K\frac{\partial V}{\partial K} + qV &= \frac{1}{2}K^2\frac{\partial^2 V}{\partial K^2}E[\sigma^2_T | X_T = K] \\
&= \frac{1}{2}K^2\int_0^\infty f^2(\overline{y})\psi(x^*, y^*, t^*; K, \overline{y}, T)d\overline{y}
\end{aligned}$$

Reformulation

- $V(K,T) \equiv V(x^*,y^*,t^*;K,T)$:

$$V(K,T) = e^{-q(T-t^*)}\left(x^* - e^{-(r-q)(T-t^*)}K\right)^+$$
$$+\int_{t^*}^{T}\int_0^\infty \frac{K^2}{2}e^{(-2r+q)(T-\tau)}f^2(\overline{y})\psi(Ke^{-(r-q)(T-\tau)},\overline{y},\tau)d\overline{y}d\tau$$

- $\psi(K,\overline{y},T) \equiv \psi(x^*,y^*,t^*;K,\overline{y},T)$:

$$\begin{aligned} \mathcal{L}^*\psi &= 0 \text{ in } K>0, \overline{y}>0, T>t^* \\ \psi|_{T=t^*} &= \delta(x^*-K)\delta(y^*-\overline{y}) \end{aligned}$$

- To determine a triple $\psi(\cdot,\cdot,\cdot)$, $V(\cdot,\cdot)$, and $b(\cdot,\cdot)$ s.t.

$$V(K_i,T_j) = V^*(K_i,T_j) \text{ for all } i,j.$$

A Tikhonov Regularization Approach

- For illustration we assume
  - $\beta(\cdot,\cdot)$ is known;
  - we aim to identify $b = b(y)$; and $V^*(K, T_1)$ for all K are given.
- We minimize

$$J(b(\cdot)) = \frac{\alpha}{2}\|\nabla b\|^2_{L^2(\mathbb{R}^+)} + \frac{1}{2}\|V(\cdot, T_1; b(\cdot)) - V^*(\cdot, T_1)\|^2_{L^2(R^+)}$$

where $\alpha > 0$ is a regularization parameter,

$$V(K, T_1; b) = e^{-q(T_1 - t^*)}\left(x^* - e^{-(r-q)(T_1-t^*)}K\right)^+$$
$$+ \int_{t^*}^{T_1}\int_0^\infty \frac{K^2}{2} e^{(-2r+q)(T_1-\tau)} f^2(\overline{y})\psi(Ke^{-(r-q)(T_1-\tau)}, \overline{y}, \tau; b)\, d\overline{y}\, d\tau$$

Necessary Condition for Optimal Solution

- Let $b(\cdot) \in H^1_{loc}(R^+)$ be the optimal solution, then

$$\frac{d}{d\lambda} J(b + \lambda w)|_{\lambda=0} = 0 \quad \text{for } w \in H^1_{loc}(R^+)$$

- Denote $b^\lambda(\cdot) = b(\cdot) + \lambda w(\cdot)$,

$$\alpha \int_0^\infty b_{\overline{y}} w_{\overline{y}} d\overline{y} + \int_0^\infty [V(K, T_1) - V^*(K, T_1)] \frac{d}{d\lambda} V(K, T_1; b^\lambda)|_{\lambda=0} dK = 0$$

Continued (I)

Consider

$$\int_0^\infty [V(K,T_1) - V^*(K,T_1)] \frac{d}{d\lambda} V(K,T_1;b^\lambda)|_{\lambda=0} dK$$

- $$\frac{d}{d\lambda} V(K,T_1;b^\lambda)|_{\lambda=0} = \int_{t^*}^{T_1} \int_0^\infty \frac{K^2}{2} e^{(-2r+q)(T_1-\tau)} f^2(\overline{y}) \frac{d}{d\lambda} \psi(Ke^{-(r-q)(T_1-\tau)}, \overline{y}, \tau; b^\lambda)|_{\lambda=0} d\overline{y} d\tau$$
- Denote $\eta(K,\overline{y},T) = \frac{d}{d\lambda}\psi(K,\overline{y},T;b^\lambda)|_{\lambda=0}$, which satisfies

$$\begin{cases} L^*\eta = -(\psi\omega)_{\overline{y}} & K>0, \overline{y}>0, T>t^* \\ \eta|_{T=t^*} = 0 \end{cases}$$

Continued (II)

When $T_1 - t^*$ is small,

$$-b_{\overline{y}\overline{y}} + \frac{f(\overline{y})f'(\overline{y})}{2\alpha}\int_0^\infty K^2[V(K,T_1;b) - V^*(K,T_1)]\psi(K,\overline{y},T_1;b)dK = 0$$

where

$$V(K,T_1;b) = e^{-q(T_1-t^*)}\left(x^* - e^{-(r-q)(T_1-t^*)}K\right)^+ + \int_{t^*}^{T_1}\int_0^\infty \frac{K^2}{2}e^{(-2r+q)(T_1-\tau)}f^2(\overline{y})\psi(Ke^{-(r-q)(T_1-\tau)},\overline{y},\tau;b)d\overline{y}d\tau$$

and $\psi(K,\overline{y},T;b)$ satisfies

$$\begin{aligned} \mathcal{L}^*\psi &= 0 \quad K>0,\overline{y}>0,T>t^* \\ \psi(K,\overline{y},t^*) &= \delta(x^*-K)\delta(y^*-\overline{y}) \end{aligned}$$

Algorithm: Gradient Descent

Introducing a “false” time s to find a stationary solution

$$\begin{aligned} b_s &= b_{\overline{yy}} \\ &- \frac{f(\overline{y})f'(\overline{y})}{2\alpha}\int_0^\infty K^2[V(K,T_1;b)-V^*(K,T_1)]\psi(K,\overline{y},T_1;b)dK \\ &= -F(b,V,\psi) \end{aligned}$$

The computation procedure

1. Set $n=0$, give an initial guess $b^0(\overline{y})$ and step size Δs.
2. Solve for $\psi(K,\overline{y},T;b^n)$ and $V(K,T;b^n)$.
3. Gradient descent: $b^{n+1}=b^n-\Delta sF(b^n,V,\psi)$.
4. If $||b^{n+1}(\overline{y})-b^n(\overline{y})||<\varepsilon$, stop. Otherwise set $n=n+1$ and go to Step 2.

Computation Domain and Boundary Conditions

$$
\begin{aligned}
b_s &= b_{\overline{y}\overline{y}} \\
&- \frac{f(\overline{y})f'(\overline{y})}{2\alpha}\int_0^\infty K^2[V(K,T_1;b) - V^*(K,T_1)]\psi(K,\overline{y},T_1;b)dK
\end{aligned}
$$

- We confine ourselves to a bounded domain $[0, \overline{y}_u]$.
- Boundary conditions
  - Dirichlet boundary conditions: b is given at $0, \overline{y}_u$.
  - Neumann boundary conditions: $b_{\overline{y}}$ is given at $0, \overline{y}_u$.

The time dependent case: $b = b(\overline{y}, T)$

Given $b(\cdot, T_i)$, $i = 1, 2, .., n$,

$$\begin{aligned} & \frac{\alpha_0}{\alpha} \frac{b(\cdot, T_{n+1}) - b(\cdot, T_n)}{\Delta T} - b_{\overline{yy}}(\cdot, T_{n+1}) \\ + \quad & \frac{f(\overline{y}) f'(\overline{y})}{2\alpha} \int_0^\infty K^2 [V(K, T_{n+1}; b(\cdot, T_{n+1})) - V^*(K, T_{n+1})] \\ & \qquad \psi(K, \overline{y}, T_{n+1}; b(\cdot, T_{n+1})) dK = 0 \end{aligned}$$

Calibrating $b = b(\cdot,\cdot)$ and $\beta(\cdot,\cdot)$

- We can similarly derive the necessary condition for $\beta(\cdot,\cdot)$
- Then we can calibrate $b = b(\cdot,\cdot)$ and $\beta(\cdot,\cdot)$ together.

Numerical Results

- $\sigma_t = f(Y_t) = \sqrt{Y_t}$:

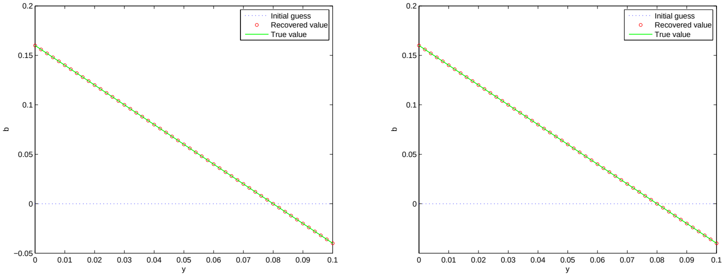
$$dY_t = b(Y_t, t)dt + \gamma\sqrt{Y_t}dW_t^1$$

- Default parameter values:

$$x^* = 1;\ y^* = 0.02;\ t^* = 0;\ T_n - T_{n-1} = \frac{1}{12};$$
$$r = 0.02;\ q = 0;\ \gamma = 0.1;\ \rho = 0.5;$$
$$\overline{y}_u = 0.1;\ \alpha = 5 * 10^{-5};\ \alpha_0 \ll \alpha$$

Test One

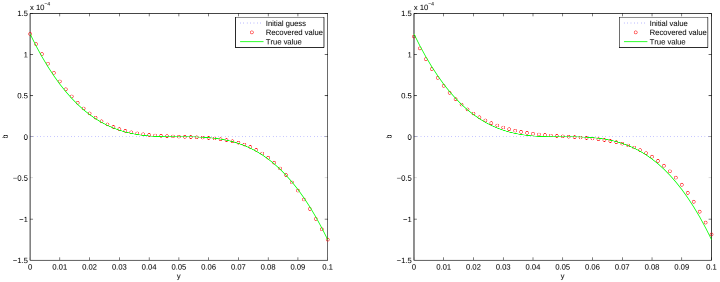
True value $b(\overline{y}) = 2(0.08 - \overline{y})$; Initial guess $b^0 = 0$



Left with *true* Dirichlet cond.; Right with *true* Neumann cond.

Test Two

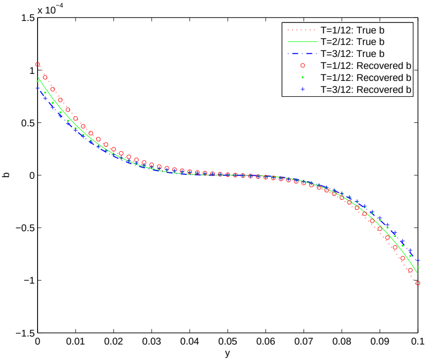
True value $b(\overline{y}) = -(\overline{y} - 0.05)^3$; Initial guess $b^0 = 0$



Left with *true* Dirichlet cond.; Right with *true* Neumann cond.

Test Three: The Time Dependent Case

True value: $b(\overline{y}, T) = -(\overline{y} - 0.05)^3/(1 + 2T)$; initial guess $b^0 = 0$



With Neumann cond.

Market Tests

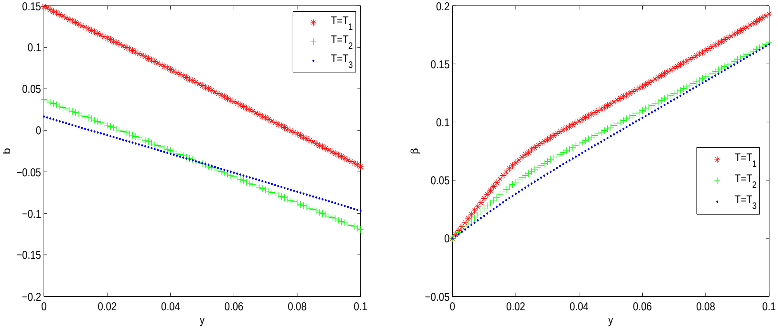


Figure : $t^* = 25/03/2013$, with maturities $T_1 = 20/04/2013$, $T_2 = 18/05/2013$, and $T_3 = 22/06/2013$.

Market Tests (continued)

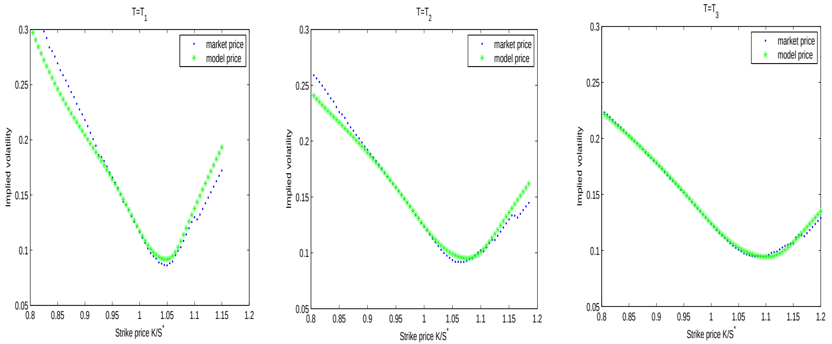


Figure : Implied volatilities computed from the market price (dots) and the calibrated model price (stars) at $t^* = 25/03/2013$, with maturities $T_1 = 20/04/2013$, $T_2 = 18/05/2013$, and $T_3 = 22/06/2013$.

Conclusion

- Formulation as a standard inverse problem of partial differential equations.
- We solve the inverse problem in terms of Tikhonov regularization. An efficient algorithm is given.
- Numerical results and market tests are presented to demonstrate the efficiency of our numerical algorithm.

Extension and Future Work

- Recovering $\rho(\cdot)$, $b(\cdot,\cdot)$, $\beta(\cdot,\cdot)$ simultaneously.
- Using elastic volatility:

$$\begin{aligned} dX_t &= (r-q)X_t dt + \sigma_t X_t^{\alpha} dW_t^0 \\ \sigma_t &= f(Y_t) \\ dY_t &= b(Y_t,t)dt + \beta(Y_t,t)dW_t^1 \end{aligned}$$

or more generally $b = b(X_t, Y_t, t)$, $\beta = \beta(X_t, Y_t, t)$.