On the general structure of optimal investment and consumption under small transaction costs

heuristics and rigorous results

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Angers, September 3, 2013

Outline

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The setup

Itô process with proportional transaction costs

- Risky asset:

$$dS_t = b_t^S dt + \sigma_t^S dW_t$$

- Wealth process:

$$X_t^\varepsilon(\psi^\varepsilon, k^\varepsilon) = x + \int_0^t \psi_s^\varepsilon dS_s - \int_0^t k_s^\varepsilon ds + \Psi_t - \int_0^t \varepsilon_s d\|\psi^\varepsilon\|_s,$$

  where
  - ψ_t^ε trading strategy (= number of shares),
  - k_t^ε consumption rate,
  - x initial endowment,
  - Ψ_t random endowment stream, e.g. $\Psi_t = 0$
  - $\varepsilon_t = \varepsilon \mathcal{E}_t$ proportional transaction costs, e.g. $\mathcal{E}_t = S_t$

- ε is supposed to be "small."

Maximising expected utility

under small proportional transaction costs

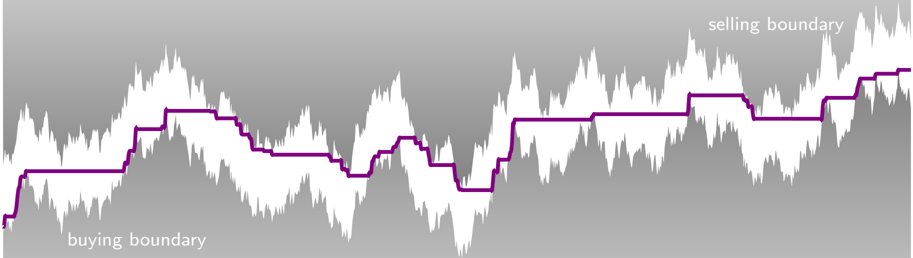
- Goal:

$$\max_{(\psi^\varepsilon, k^\varepsilon)} E\left[\int_0^T u_1(t, k_t^\varepsilon)dt + u_2(X_T^\varepsilon(\psi^\varepsilon, k^\varepsilon))\right]$$

  with possibly random utility functions $u_1(t, \cdot), u_2(\cdot)$

- More precisely: determine leading-order correction to frictionless problem ($\varepsilon = 0$) for small costs ε

width of no-trade corridor: $\Delta \mathrm{NT}_t = \sqrt[3]{12 R_t \frac{d\langle \varphi \rangle_t}{d\langle S \rangle_t} \varepsilon_t}$



certainty equivalent loss: $\Delta \mathrm{CE} = E^Q \left[\int_0^T \frac{(\Delta \mathrm{NT}_t)^2}{8 R_t} d\langle S \rangle_t \right]$

Known general structure

and some references

- There is a no-trade region around the frictionless optimiser.
- Do nothing while portfolio is inside no-trade region.
- Do infinitesimal trades at boundary of no-trade region.
- Width of the no-trade region is of order $\varepsilon^{1/3}$.
- (Certainty equivalent of) utility loss is of order $\varepsilon^{2/3}$.
- Some references: Magill and Constantinides (1976), Constantinides (1986), Dumas and Luciano (1991), Taksar et al. (1988), Davis and Norman (1990), Shreve and Soner (1994) and many more
- References on small costs: Shreve and Soner (1994), Whalley and Willmott (1997), Janeček and Shreve (2004), Goodman and Ostrov (2010), Bichuch (2011), K. and Muhle-Karbe (2012), Martin (2012), Soner and Touzi (2012), Possamaï et al. (2012), and more

Outline

Leading-order asymptotics

- Asymptotic no-trade region: $[\overline{\mathrm{NT}}_t - \Delta\mathrm{NT}_t, \overline{\mathrm{NT}}_t + \Delta\mathrm{NT}_t]$,

$$\overline{\mathrm{NT}}_t = \varphi_t + \varphi_t'(X_t^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon) - X_t(\varphi, \kappa)),$$

$$\boxed{\Delta NT_t = \sqrt[3]{\frac{3R_t}{2}\frac{d\langle\varphi\rangle_t}{d\langle S\rangle_t}\varepsilon_t}}$$

- Asymptotic consumption rate:

$$\kappa_t^\varepsilon = \kappa_t + \kappa_t'(X_t^\varepsilon(\varphi^\varepsilon, \kappa^\varepsilon) - X_t(\varphi, \kappa)),$$

where

- φ_t' derivative of frictionless optimal portfolio wrt. time-t wealth,
- κ_t' derivative of frictionless optimal consumption wrt. time-t wealth,
- R_t indirect risk tolerance of the frictionless problem

Indirect risk tolerance R_t

of the frictionless problem

- $$R_t := -\frac{U'(t,X_t)}{U''(t,X_t)}$$

  for indirect utility

  $$U(t,x) := \sup_{(\psi_t,k_t)_{t\in[t,T]}} E_t\left[\int_t^T u_1(s,k_s)ds + u_2\left(x + \int_t^T \psi_s dS_s - \int_t^T k_s ds + \Psi_t\right)\right]$$

  Compare Kramkov and Sîrbu (2006), Soner and Touzi (2012)
- R yields φ', κ' via

  $$\varphi'_t = \frac{d\langle R,S\rangle_t}{R_t d\langle S\rangle_t}, \quad \kappa'_t = \frac{r_t}{R_t}$$

  with direct risk tolerance

  $$r_t := -\frac{u_1'(t,\kappa_t)}{u_1''(t,\kappa_t)}.$$

Indirect risk tolerance R_t

as solution to a BSDE

R is solution to quadratic BSDE

$$dR_t = \left(\frac{\zeta_t^\top \zeta_t}{R_t} - \frac{(\sigma_t^\top \zeta_t)^2}{R_t \sigma_t^\top \sigma_t} - r_t \right) dt + \zeta_t dW_t^Q, \quad R_T = -\frac{u_2'(X_T)}{u_2''(X_T)}$$

if

- Q dual minimiser for frictionless utility maximisation problem,
- filtration generated by d-dimensional Brownian motion W^Q,
- $dS_t = \sigma_t dW_t^Q$.

Indirect risk tolerance R_t

for standard utility functions without random endowment

- Exponential utility $u_2(x) = -e^{-px}$ and $u_1(t,x) = -e^{\delta(T-t)}e^{-px}$:

$$R_t = \frac{1+T-t}{p},$$
$$\varphi'_t = 0, \quad \kappa'_t = \frac{1}{1+T-t},$$
$$\overline{\mathrm{NT}}_t = \varphi_t$$

- Power utility $u_2(x) = \frac{x^{1-p}}{1-p}$ and $u_1(t,x) = -e^{\delta(T-t)}\frac{x^{1-p}}{1-p}$:

$$R_t = \frac{X_t}{p},$$
$$\varphi'_t = \frac{\varphi_t}{X_t}, \quad \kappa'_t = 1,$$

no-trade region centered in relative terms.

Leading-order asymptotics

continued

- Asymptotic certainty equivalent of utility loss

$$E^Q\left[\int_0^T \frac{(\Delta \mathrm{NT}_t)^2}{2R_t} d\langle S\rangle_t\right]$$

with dual minimiser Q of frictionless problem ($2/3$ loss due to transaction costs, $1/3$ loss due to displacement, cf. Rogers 2004)

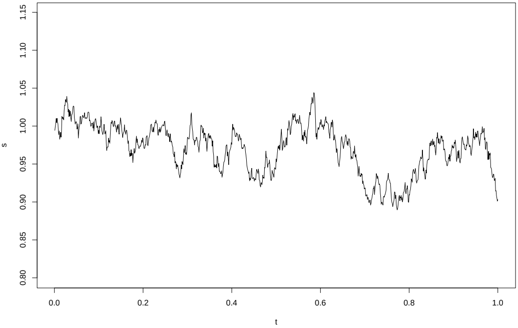
- Implied trading volume

$$\|\varphi^\varepsilon\|_T \sim \int_0^T \sqrt[3]{\frac{16}{3R_t}\left(\frac{d\langle\varphi\rangle_t}{d\langle S\rangle_t}\right)\frac{1}{\varepsilon_t}} d\langle S\rangle_t$$

Outline

Shadow prices approach

(Cvitanić and Karatzas 1996, Loewenstein 2000, . . .)



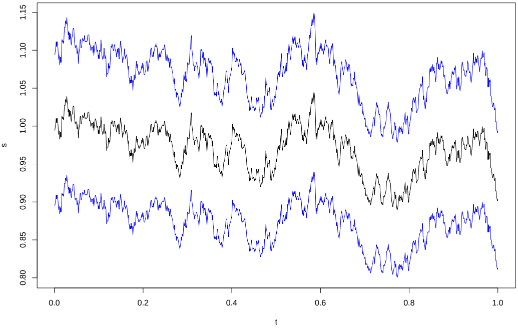
- General priciple:
  portfolio optimization in market with transaction costs

  $$\Longleftrightarrow$$

  portfolio optimization without transaction costs for some shadow process within bid-ask bounds

Shadow prices approach

(Cvitanić and Karatzas 1996, Loewenstein 2000, . . .)



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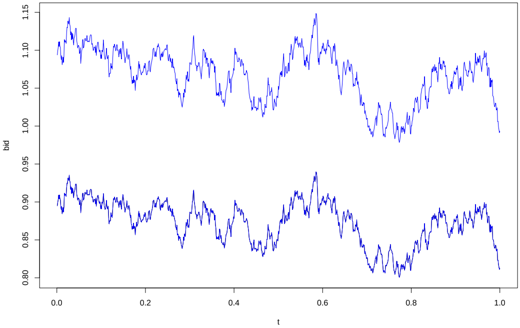

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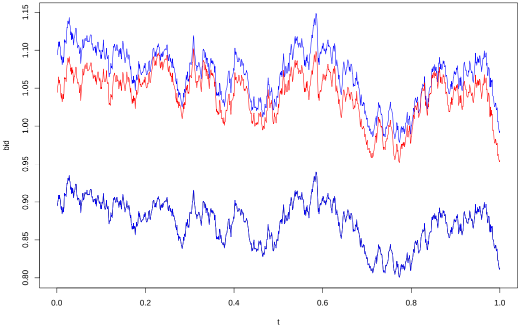

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Shadow price approach

in the terminal wealth case $u_1 = 0$ without random endowment

Look for

- strategy φ^ε_t,
- price process $\widetilde{S}^\varepsilon_t \in [\underline{S}_t, \overline{S}_t]$,
- process Z^ε_t,

which satisfy

- $Z^\varepsilon_T = u_2'(x + \varphi^\varepsilon \bullet \widetilde{S}^\varepsilon_T)$,
- Z^ε martingale,
- $Z^\varepsilon \widetilde{S}^\varepsilon$ martingale,
- φ^ε changes only when $\widetilde{S}^\varepsilon_t \in \{\underline{S}_t, \overline{S}_t\}$.

Then φ^ε is the optimal portfolio.

Optimality of φ^ε

informally

For any competitor ψ we have

$$
\begin{aligned}
E[u(V_T(\psi))] &\le E[u(x + \psi \bullet \widetilde{S}^\varepsilon_T)] \\
&\le E[u(x + \varphi^\varepsilon \bullet \widetilde{S}^\varepsilon_T)) + E\Big[u'(x + \varphi^\varepsilon \bullet \widetilde{S}^\varepsilon_T)((\psi - \varphi^\varepsilon) \bullet \widetilde{S}^\varepsilon_T)\Big] \\
&= E[u(V_T(\varphi^\varepsilon))] + E\Big[Z^\varepsilon_T((\psi - \varphi^\varepsilon) \bullet \widetilde{S}^\varepsilon_T)\Big] \\
&= E[u(V_T(\varphi^\varepsilon))].
\end{aligned}
$$

(Second inequality follows from $u(y) \le u(x) + u'(x)(y - x)$.)

Approximate solution

Look for

- strategy φ^ε,
- price process $\widetilde{S}^\varepsilon_t \in [\underline{S}_t, \overline{S}_t]$,
- process Z^ε,

which satisfy

- $Z^\varepsilon_T = u'(x + \varphi^\varepsilon \bullet \widetilde{S}^\varepsilon_T) + O(\varepsilon)$,
- Z^ε has drift $o(\varepsilon^{2/3})$,
- $Z^\varepsilon \widetilde{S}^\varepsilon$ has drift $O(\varepsilon^{2/3})$,
- φ^ε changes only when $\widetilde{S}^\varepsilon_t \in \{\underline{S}_t, \overline{S}_t\}$.

Then φ^ε is optimal to the leading order.

Approximate optimality of φ^ε

informally

For any competitor ψ^ε with $\psi^\varepsilon - \varphi = o(1)$ we have

$$
\begin{aligned}
&E[u(V_T(\psi^\varepsilon))] \leq E[u(x + \psi^\varepsilon \bullet \widetilde{S}_T^\varepsilon)] \\
&\quad\leq \quad E[u(x + \varphi^\varepsilon \bullet \widetilde{S}_T^\varepsilon)] + E\Big[u'(x + \varphi^\varepsilon \bullet \widetilde{S}_T^\varepsilon)((\psi^\varepsilon - \varphi^\varepsilon) \bullet \widetilde{S}_T^\varepsilon)\Big] \\
&\quad= \quad E[u(V_T(\varphi^\varepsilon))] + E\Big[Z_T^\varepsilon((\psi^\varepsilon - \varphi^\varepsilon) \bullet \widetilde{S}_T^\varepsilon)\Big] + o(\varepsilon^{2/3}) \\
&\quad= \quad E[u(V_T(\varphi^\varepsilon))] + o(\varepsilon^{2/3}).
\end{aligned}
$$

Compare with

$$E[u(V_T(\varphi^\varepsilon))] - E[u(x + \varphi \bullet \widetilde{S}_T)] = O(\varepsilon^{2/3}).$$

Outline

A rigorous theorem

Setup

- Itô process

$$dS_t = b_t^S dt + \sigma_t^S dW_t$$

- Exponential utility $u(x) = -e^{-px}$ of terminal wealth
- Transaction costs $\varepsilon_t = \varepsilon S_t$
- Maximise

$$CE(X_T(\psi)) = -\frac{1}{p} \log E\left[\exp(-pX_T(\psi))\right]$$

A rigorous theorem

Assumptions (sketched)

- There exists EMM for S with finite relative extropy.
  $\rightsquigarrow$ MEMM Q and frictionless optimiser φ exist.
- φ and $\frac{d\langle\varphi\rangle}{d\langle S\rangle}$ are Itô processes.
- A number of integrability conditions, mostly of the kind

$$E_Q\left[\sup_{t\in[0,T]} g_t^n\right] < \infty, n \in \mathbb{N}$$

for some processes g

A rigorous theorem

Statement

- Define ΔNT as before.
- There exists $\varphi^\varepsilon = \varphi + \Delta\varphi = \varphi^{\varepsilon\uparrow} - \varphi^{\varepsilon\downarrow}$ with values in $[\varphi - \Delta NT, \varphi - \Delta NT]$, where $\varphi^{\varepsilon\uparrow}, \varphi^{\varepsilon\downarrow}$ increase only at the boundary.
- Set

$$\tau^\varepsilon := \inf\left\{t \in [0,T] : |X_t(\varphi^\varepsilon) - (x + \varphi \bullet S_t)| > 1 \text{ or } |X_t(\varphi^\varepsilon)| > \varepsilon^{-4/3}\right\}$$

- Then $\varphi^\varepsilon 1_{[\![0,\tau^\varepsilon]\!]}$ is optimal to the leading order with certainty equivalent loss as stated earlier.

A rigorous theorem

Idea of the proof (cf. Henderson 2002, Kramkov & Sirbu 2006)

1. Existence of the process $\Delta\varphi$ as solution to a Skorohod problem with reflection.
2. Definition of $\Delta\tilde{S}$ as a function of $\Delta\varphi$.
3. Leading-order approximation of the corresponding expected utility by Taylor expansion.
4. Modify candidate Z^ε to an EMM $\tilde{Z}^\varepsilon$ for $\tilde{S}^\varepsilon$.
5. Compute leading-order approximation of Lagrange dual function at $\tilde{Z}^\varepsilon$. Observe that lower and upper bound coincide to leading order.
6. Replace $\Delta\varphi$ in expression for certainty equivalent loss by ΔNT, using the ergodic property of reflected Brownian motion.

Outline

Main conclusions

concerning leading order asymptotics

- Robust formulas for no-trade region, consumption rate, utility loss, implied trading volume
- In some sense myopic results
- Important ingredients of frictionless problem:
  - indirect risk tolerance R_t,
  - activity rate or generalized gamma $\frac{d\langle\varphi\rangle_t}{d\langle S\rangle_t}$
- $2/3$ loss due to transaction costs, $1/3$ loss due to displacement

Next steps

Things we would like to consider

- Rigorous proofs in full generality.
- Generalise results to the multivariate case.
- Study alternative cost structures.