Credit-Equity Modeling under a Latent Lévy Firm Process

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Plan of My Talk

1. Introduction: Motivation, Literature Review
   - Equity Options, Credit Modeling, Credit-Equity Models
   - CreditGrades, Latent Credit Model
2. Our Model
   - Firm Value, Equity Value
   - Regime-Switching (mid-term spread), Jumps (short-term spread)
   - CDS, Equity Option Pricing
3. Numerical Examples
   - CDS, Equity Option
4. Conclusion, Future Research

Motivation

- Recent credit crisis shows the intimate relationship between the credit and equity markets.
- For example, during the credit crisis, both CDS premiums and equity volatilities were at their historical high.
- However, until recently, the equity and credit modelings are two separate themes in the finance literature.
- The difficulty to construct the credit-equity model stems from the fact that the debt and equity possess different properties.
- Hence, new attempts are required to construct the credit-equity modeling in a unified manner.

Literature Review: Equity Options

- Mostly, based on Stochastic Differential Equations (SDEs)
  - Diffusion: Black–Scholes Model, Local Volatility Model
  - Stochastic Volatility (SV): Heston (1993), etc.
  - Jump Diffusion: Merton (1976), Kou (2002), etc.
  - Lévy Process: Ask Professor Vostrikova
  - Regime Switching: Kijima and Yoshida (1993), Buffington and Elliott (2002), etc.
- Strength: Highly liquid markets (equity and equity options)
- Shortcoming : No mention about the issuer's (firm) credit exposure

Literature Review: Credit Modeling

1. Reduced-form approach
   - Jarrow and Turnbull (1995), Madan and Unal (1998a), Duffie and Singleton (1999), etc.
   - Strength: Analytical tractability and ability of generating a flexible and realistic term structure of credit spreads
   - Shortcoming 1: Exogenous hazard rate process
   - Shortcoming 2: Default mechanism is not related to the firm value
2. Structural approach
   - Merton (1974), Black and Cox (1976), Leland (1994), etc.
   - Strength: Economic appeal–links firm value with debt and equity
   - Shortcoming 1: Difficult to incorporate more realistic features without sacrificing tractability
   - Shortcoming 2: Difficult to price equity or equity options

Literature Review: Credit-Equity Models

1. Reduced-form approach
   - Mendoza-Arriaga et al. (2010) and the references therein
2. Structural approach
   - CreditGrades model by Finger et al. (2002) and its extensions
     - by Sepp (2006) for double-exponential jump-diffusion model
     - by Ozeki et al. (2011) for general spectrally-negative Lévy process
   - Time-changed Brownian motion approach by Hurd and Zhou (2011)
   - Latent model by Kijima et al. (2009)

Literature Review: CreditGrades

1. Ordinary structural approach
   - Consider a corporate firm that issues a debt and an equity.
   - Let D and S be the debt and equity values per share, respectively.
   - Let V be the firm value per share, so that $V = D + S$ by the basic accounting assumption.
   - V is modeled by a SDE and the default occurs when V reaches a default threshold.
   - D and S are evaluated as contingent claims written on V.
2. CreditGrades model by Finger et al. (2002)
   - D is the discounted face value of debt and S is modeled by a GBM.
   - V is given by $V = D + S$, and default is the first passage time of V.
   - Strength: Easy to implement and extend.
   - Shortcoming 1: D is irrelevant to the credit structure.
   - Shortcoming 2: Credit quality is essentially equal to the equity value.

Literature Review: Latent Credit Model

1. Latent model
   - Introduce the notion of the marker process that is observable and correlated to the actual status process (unobservable).
2. Latent structural model by Kijima et al. (2009)
   - The actual firm status is latent.
   - Debt value is given in terms of the actual firm status.
   - Equity value is obtained as a residual value as in Merton (1974).
   - Strength: Economically appealing
   - Shortcoming 1: The equity has a maturity as in Merton (1974).
   - Shortcoming 1: The pricing of equity options is very complicated.

Our Model: Overview

- Structural approach: treat the firm value as a latent variable
- Extension of Kijima et al. (2009) to include jumps (for short-term credit spread) and regime switch (for mid-term spread)
- Source of information: Equity value
- Objective: Price CDS and Equity Option with default feature under a joint framework.
- Contributions: Our model
  1. Introduces the credit status of the firm into the equity process.
  2. Serves as a theoretical support to the existing empirical analyses on the explanatory power of equity's historical and option-implied volatilities to the CDS spread variation.
  3. Has a flexibility in explaining both the short-term and mid-term behaviors of the credit spread and implied volatility curves.

Our Model: Firm value process

- A_t: Actual firm value at time t where

$$A_t = \exp(X_t), \quad t \geq 0$$

- A_t is latent, i.e. unobservable and non-tradable.
- Nature of default: Default epoch τ is defined by

$$\tau = \inf\{t \geq 0 : A_t \leq \Gamma\} = \inf\{t \geq 0 : X_t \leq L\}$$

for some $\Gamma = e^L$ (default barrier).

- Remark: Easy to extend to include a stochastic boundary.

Our Model: Equity value process

- S_t: Equity value of the firm at time t
- S_t is observable to investors and tradable.
- Let $Y_t = \log S_t$, and assume that (for each regime)

$$Y_t = \rho X_t + Z_t$$

- Z_t: Non-firm specific shocks, independent of X_t (given each regime).
- ρ: The impact factor of firm's credit exposure on equity

Regime-Switching

- Introduce the regime-switching for the mid-term spread.
- Let $\{J_t : t \geq 0\}$ be a Markov chain on state space E. .
- E is finite and contains d elements, i.e., $E = \{1, 2, \ldots, d\}$.
- Let $\mathbf{Q}$ be the intensity matrix of J_t with respect to the Lebesgue measure, i.e.,

$$\mathbf{Q} = \{q_{ij}\}_{i,j \in E}$$

where

$$q_{ii} = -\sum_{i \neq j} q_{ij}$$

Model of Log-Firm Value

- Let $X_t = \log A_t$ be defined by

$$
\begin{aligned}
X_t &= \int_0^t b^X(J_s)\mathrm{d}s + \int_0^t \sigma^X(J_s)\mathrm{d}W_s^X \\
&\quad + \sum_{j \in E} \int_0^t 1_{\{J_s = j\}} \mathrm{d}N_s^X(j)
\end{aligned}
$$

  where, given $J_t = j \in E$, $b^X(J_t) \equiv b_j^X$ is a drift, $\sigma^X(J_t) \equiv \sigma_j^X$ is a volatility, and $\{N_t^X(j) : t \geq 0\}$ is a compound Poisson process.

- $N_t^X(j)$ has arrival rate λ_j^X and double-exponential jumps Y_j^X with distribution $\nu_j^X(\mathrm{d}y)$, where

$$
\begin{aligned}
\nu_j^X(\mathrm{d}y) &= \lambda_j^X \Big[p_j^X \eta_{j1}^X \mathrm{e}^{-\eta_{j1}^X y} 1_{\{y \geq 0\}} \\
&\quad + (1 - p_j^X) \eta_{j2}^X \mathrm{e}^{\eta_{j2}^X y} 1_{\{y < 0\}} \Big] \mathrm{d}y
\end{aligned}
$$

Moment Generating Function of X_t

It is well known that the moment generating function (MGF) of $\boldsymbol{X_t}$, $\mathbb{E}[\exp(\boldsymbol{u}\boldsymbol{X_t})]$, is given by

$$\mathbb{E}[\exp(\boldsymbol{u}\boldsymbol{X_t})] \equiv \exp\left(\mathsf{K}^{\boldsymbol{X}}[\boldsymbol{u}]\boldsymbol{t}\right)$$

where

$$\mathsf{K}^{\boldsymbol{X}}[\boldsymbol{u}] \equiv \{\boldsymbol{\kappa}_j^{\boldsymbol{X}}(\boldsymbol{u})\}_{\text{diag}} + \mathbf{Q}$$

with

$$\boldsymbol{\kappa}_j^{\boldsymbol{X}}(\boldsymbol{u}) = \boldsymbol{b}_j^{\boldsymbol{X}}\boldsymbol{u} + \frac{(\boldsymbol{\sigma}_j^{\boldsymbol{X}}\boldsymbol{u})^2}{2} + \boldsymbol{\lambda}_j^{\boldsymbol{X}}\left(\frac{\boldsymbol{p}_j^{\boldsymbol{X}}\boldsymbol{\eta}_{j1}^{\boldsymbol{X}}}{\boldsymbol{\eta}_{j1}^{\boldsymbol{X}} - \boldsymbol{u}} + \frac{(1 - \boldsymbol{p}_j^{\boldsymbol{X}})\boldsymbol{\eta}_{j2}^{\boldsymbol{X}}}{\boldsymbol{\eta}_{j2}^{\boldsymbol{X}} + \boldsymbol{u}} - 1\right)$$

for double-exponential jumps.

Model of Non-Firm Specific Shock

- Recall that $Y_t = \log S_t$ and, for each regime, $Y_t = \rho X_t + Z_t$.
- We assume that Z_t has the following canonical representation:

$$Z_t = \int_0^t b^Z(J_s)\mathrm{d}s + \int_0^t \sigma^Z(J_s)\mathrm{d}W_s^Z + \int_0^t \int_{\mathbb{R}} y(\mu^Z(J_s) - \nu^Z(J_s))(\mathrm{d}y)\mathrm{d}s$$

  where, for $J_t = j$, $b^Z(J_t) \equiv b_j^Z$ is a drift, $\sigma^Z(J_t) \equiv \sigma_j^Z$ is a volatility, $\mu^Z(J_t) \equiv \mu_j^Z$ is a random jump measure, and $\nu^Z(J_t) \equiv \nu_j^Z$ is the compensator of μ_j^Z.
- Z_t can be a general Lévy, because it is irrelevant to default.

Moment Generating Function of Z_t

The MGF $\mathbb{E}[\exp(uZ_t)]$ is given by

$$\mathbb{E}[\exp(uZ_t)] \equiv \exp\left(\mathsf{K}^Z[u]t\right)$$

where

$$\mathsf{K}^Z[u] \equiv \{\kappa_j^Z(u)\}_{\text{diag}} + \mathbf{Q}$$

with

$$\kappa_j^Z(u) = b_j^Z u + \frac{1}{2}(\sigma_j^Z u)^2 + \int_{\mathbb{R}} (\mathrm{e}^{uy} - 1 - y\mathbf{1}_{\{|y|\leq 1\}})\nu_j^Z(\mathrm{d}y)$$

No-Arbitrage Condition

Assume $J_t = j$. The discounted process $\bar{S}_t \equiv \mathrm{e}^{-rt} S_t$ is a $\mathbb{P}$-martingale with respect to $\mathcal{F}_t$ if and only if

$$\begin{aligned} \rho b_j^X + b_j^Z &= r - \frac{1}{2}(\rho\sigma_j^X)^2 - \frac{1}{2}(\sigma_j^Z)^2 \\ &\quad - \lambda_j^X \left(\frac{p_j^X \eta_{j1}}{\eta_{j1} - \rho} + \frac{(1 - p_j^X)\eta_{j2}^X}{\eta_{j2}^X + \rho} - 1 \right) \\ &\quad - \lambda_j^Z \left(\frac{p_j^Z \eta_{j1}}{\eta_{j1} - 1} + \frac{(1 - p_j^Z)\eta_{j2}^Z}{\eta_{j2}^Z + 1} - 1 \right) \end{aligned}$$

where $r > 0$ is the risk-free interest rate.

Credit Default Swap

- Standard CDS premium formula: For $J_0 = i \in E$,

$$
\begin{aligned}
c_T^{(i)} &= (1-R)\frac{\int_0^T e^{-rt}\mathrm{d}\mathbb{P}_i(\tau \le t)}{\int_0^T e^{-rt}\mathbb{P}_i(\tau > t)\mathrm{d}t} \\
&= (1-R)r\frac{\mathbb{E}_i\left[\mathrm{e}^{-r\tau}\mathbf{1}_{\{\tau<T\}}\right]}{1-\mathbb{E}_i[\mathrm{e}^{-r\tau}\mathbf{1}_{\{\tau<T\}}]-\mathrm{e}^{-rT}\mathbb{P}_i(\tau > T)}
\end{aligned}
$$

  where R is the recovery rate and r is the risk-free interest rate.

- Hence, we need to evaluate $\mathbb{E}_i\left[\mathrm{e}^{-r\tau}\mathbf{1}_{\{\tau<T\}}\right]$, $\imath \in E$.
- Following a similar discussion to Kijima and Siu (2013), these values are obtained as a solution of a linear equation, when jumps are double-exponential.

Short-Term Credit Spread

Lemma

Denote $x = -\log(\frac{L}{A_0})$ *and* $J_0 = i$. *Then,*

$$\lim_{T \downarrow 0} c_T^{(i)} = r(1-R)\nu_i^X((-\infty, x])$$

where ν_i *denotes the Lévy measure under regime* i.

Implication: Regime-switching Brownian motion alone CANNOT produce non-zero credit spread at $T \downarrow 0$!

$\Rightarrow$ We need jumps for plausible short-term spreads.

Long-Term Credit Spread

Lemma

Assume $\mathbb{P}[\tau < \infty] = 1$ *and* $J_0 = i$. *Then,*

$$\lim_{T\to\infty} c_T^{(i)} = (1-R)r\frac{\mathbb{E}_{\Pi}[e^{-r\tau}]}{1-\mathbb{E}_{\Pi}[e^{-r\tau}]},$$

where Π *denotes the stationary distribution of* J_t.

Implication: Impact of the regime-switching factor appears in the medium part of the CDS term structure!

CDS Premium

Corollary

In our model, the CDS premium c is given by

$$c_T^{(i)} = (1-R)r\frac{P_2^{RS}}{1-P_2^{RS}-\mathrm{e}^{-rT}P_1^{RS}}$$

where $J_0 = i$,

$$P_1^{RS} = \mathcal{L}_T^{-1}\left(\frac{1}{a}-\frac{1}{a}\mathbb{E}_i[\mathrm{e}^{a\tau}; J_\tau]\right)$$

and

$$P_2^{RS} = \mathcal{L}_T^{-1}\left(\frac{1}{a}\mathbb{E}_i[\mathrm{e}^{-(r+a)\tau}; J_\tau]\right)$$

Here, $\mathcal{L}^{-1}$ denotes the inverse Laplace transform.

Numerical Results

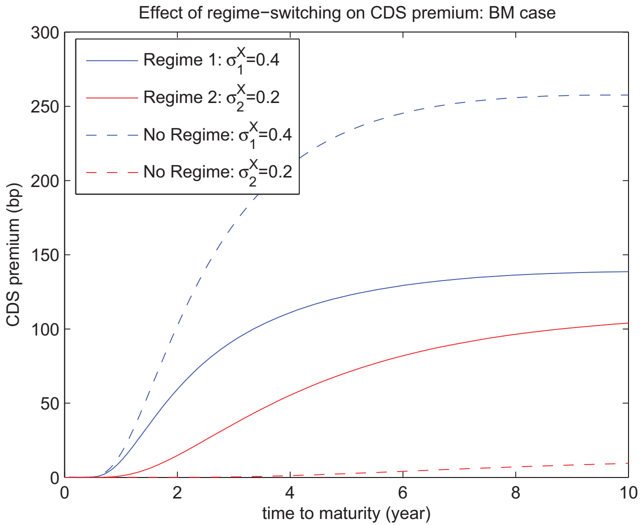
- Model Parameters for $\boldsymbol{X_t}$:

Base Parameters				
A_0	T	r	L	R
100	1	0.05	30	0.5

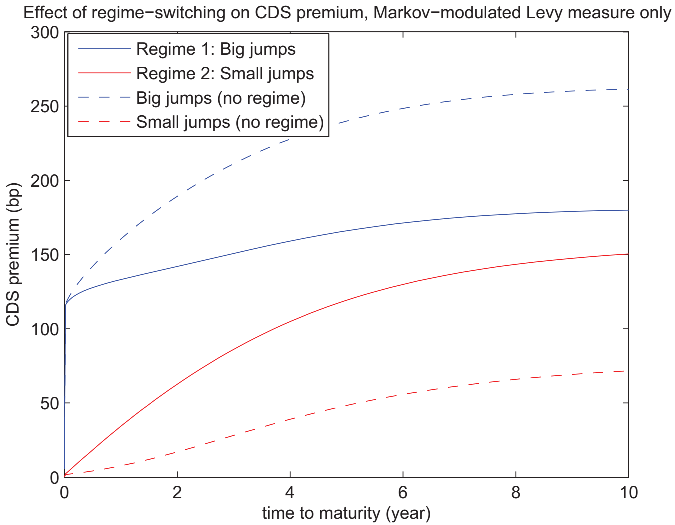
Regime 1						
b_1^X	σ_1^X	η_{11}^X	η_{12}^X	p_1^X	λ_1^X	q_1
0.05	0.4	3	2	0.5	0.5	0.5
Regime 2						
b_2^X	σ_2^X	η_{21}^X	η_{22}^X	p_2^X	λ_2^X	q_2
0.05	0.2	8	6	0.6	1	0.5

- Regime 1 (Regime 2, resp.) is of high (low) volatility and bigger (smaller) jumps.

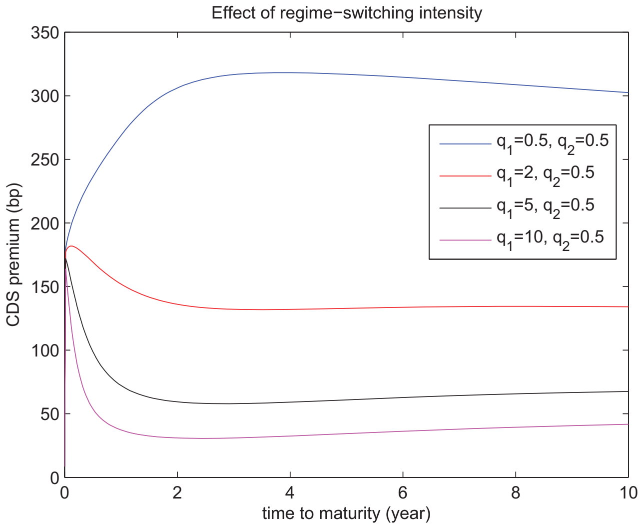
Regime-Switching Factor: BM only



Regime-Switching, Jump-Diffusion



Effect of Regime-Switching Intensity: $q_2 = 0.5$



Summary of Numerical Examples

- Hump and inverted-hump shapes of the CDS curves can be constructed by changing the regime-switching intensities of J_t.
  - Possible explanation: When buying CDS, investors are concerned with (1) current state of the firm, and (2) persistence of a firm staying in one particular economic/credit regime.
- Short-term spreads become more realistic by the jump effects.
- Introduction of regime-switching, jump-diffusion results in more flexible CDS term structures!

Equity Option

- Recall that $S_t = \exp(Y_t)$ with $Y_t = \rho X_t + Z_t$
- The call option price written on S under $\{\tau > T\}$ is given by

$$\begin{aligned} C(S,K,T) &= \mathbb{E}[\mathrm{e}^{-rT}(S_T - K)^+ 1_{\{\tau>T\}}] \\ &= \mathbb{E}[\mathrm{e}^{-rT}(S_T - K)^+] \\ &\qquad - \mathbb{E}\left[\mathrm{e}^{-rT}(S_T - K)^+ 1_{\{\tau\leq T\}}\right] \end{aligned}$$

- Hence, equivalently,

Defaultable call = Non-defaultable call − Down-and-in call

Equity Option Price

Theorem

The double Laplace transform of $\mathbb{E}[\mathrm{e}^{-rT}(S_T - K)^+ 1_{\{\tau \le T\}}]$ *with respect to* $k = \log K$ *and* T *is obtained as*

$$\begin{aligned} &\mathcal{L}_{\xi,\beta}(\mathbb{E}[\mathrm{e}^{-rT}(S_T - K)^+ 1_{\{\tau \le T\}}]) \\ = \; &\frac{S_0^{\xi+1}}{\xi(\xi+1)} \sum_j \tilde{\mathbb{E}}_i \left[e^{-((\beta+r)-\kappa_j^{Z}(\xi+1))\tau + (\xi+1)\rho X_\tau} 1_{\{J_\tau = j\}} \right] \\ \times \; &\sum_n \left((r+\beta)\mathbb{I} - \left(\left\{ \kappa_j^{Z}(\xi+1) + \kappa_j^{X}(\rho(\xi+1)) \right\}_{diag} + \mathrm{Q} \right) \right)^{-1}_{jn} \end{aligned}$$

where $\tilde{\mathbb{E}}_i$ *is the expectation under which* Z_t *is taken as the numeraire.*

Numerical Results

Model Parameters for X_t:

Base Parameters					
S_0	K	A_0	T	r	L
100	90	100	1	0.05	30

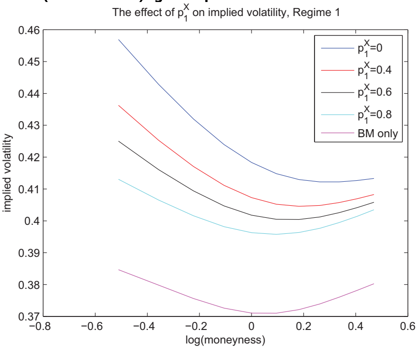
Regime 1						
b_1^X	σ_1^X	η_{11}^X	η_{12}^X	p_1^X	λ_1^X	q_1
0.05	0.4	10	4	0.4	0.5	0.5
Regime 2						
b_2^X	σ_2^X	η_{21}^X	η_{22}^X	p_2^X	λ_2^X	q_2
0.05	0.1	20	10	0.4	1	0.5

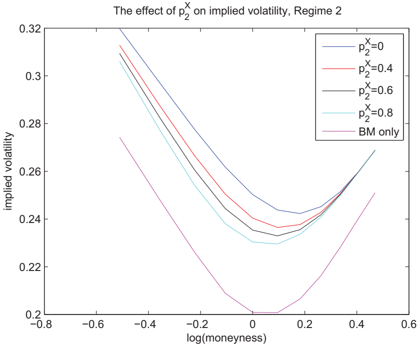
Model Parameters for Z_t (double-exponential for simplicity):

Regime 1				
σ_1^Z	η_{11}^Z	η_{12}^Z	p_1^Z	λ_1^Z
0.1	40	40	0.6	3
Regime 2				
σ_2^Z	η_{21}^Z	η_{22}^Z	p_2^Z	λ_2^Z
0.1	60	60	0.4	4

Impact of Jump Factor

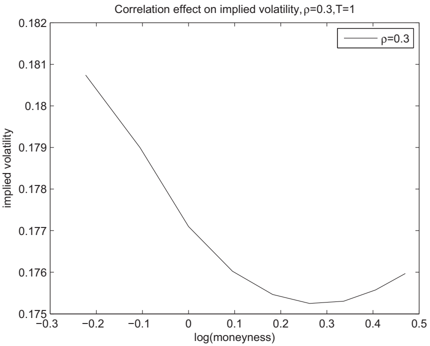
- Regime-switching BM produces symmetric smiles.
- Negative skewness is a common feature found in equity markets.
- The negative skewness is more pronounced as the probability of upper jumps p_i^X decreases, since the probability of default is decreased.
- Regime 1 (Regime 2, resp.) is of high (low) volatility and bigger (smaller) jumps.

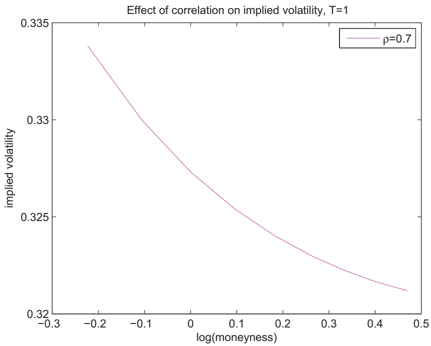




Effect of ρ without Regime-Switch

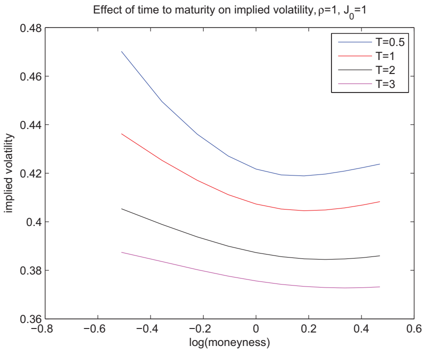
- The curvature of IV curve decreases with increasing correlation ρ.
- That is, the increase in ρ augments the negative skewness of IV.
- The negative skewness reflects the credit nature on the equity.

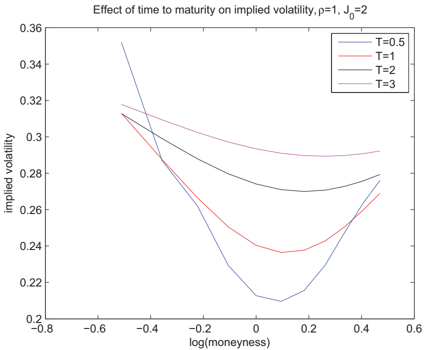




Time Effect

- Volatility curves flatten with increasing maturity T.
- In Regime 1, the IV curve moves downward as it flattens, whereas it elevates as its curvature decreases in Regime 2.
- Of course, they converge to coincide as $T \to \infty$.





Summary of Numerical Examples

- Regime-switching and jump factors play a significant role in the equity option with default feature.
- In particular, the IV curve of the high regime decreases, while it increases under the low regime, as the switching-intensity or the maturity lengthens.
- The default probability contributes to the negative skewness of IV.
- However, the degree of negative skewness is limited, in comparison with the reduced-form credit-equity model in Carr and Wu (2010).
- The assumption of independent and stationary increments of Lévy processes makes it inflexible in capturing the IV observed in the market (see Konikov and Madan, 2002).

Conclusion

- Increasing evidence of the linkage between the equity and credit aspects of a corporate firm demands a unified equity-credit model.
- We propose one approach to the problem: *Latent* structural model.
- Extend Kijima et al. (2009) to include jumps and regime-switch.
- Application: Price CDS and equity option under one framework.
- Strength: Separate jumps and regime-switch effects.
- Strength: More flexible CDS term structures and IV surfaces.
- Strength: Clarify the role of impact factor $\boldsymbol{\rho}$ to the skewness of volatility smiles.
- Numerical scheme: Inverse Laplace transform is very easy and stable.

Future Research

- Need to develop a calibration scheme.
- Want to extract credit quality (e.g., distance to default) under the physical measure from the marker process (i.e., equity value process).
- These can lead to more empirical works.
- Extend the model to include the Heston-type SV (to increase the negative skewness).
- The pricing of equity default swap, which has both the equity and credit components of a firm.

Thank You for Your Attention