Asymptotics for sums and differences of log-normal random variables

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Outline

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Introduction

Consider the random variable X^β given by

$$X^\beta = \sum_{k=1}^{n} \beta_k e^{Y_k}$$

where $\beta_1, \ldots, \beta_n$ are nonzero constants and $Y = (Y_1, \cdots, Y_n)$ is a n-dimensional Gaussian random variable with the mean $\mu = (\mu_1, \cdots, \mu_n)$ and the covariance matrix $\mathfrak{B}$ with $\det \mathfrak{B} \neq 0$, whose elements are denoted by b_{ij}

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- Sums / differences of correlated log-normal random variables appear in financial mathematics, insurance, and many other domains such as, for example, signal processing.
- In finance: models for portfolios and market indices.
- In insurance: aggregate loss from a large number of claims.

Our aims

Characterize the tail behavior of the distribution function $\mathbb{P}[X^\beta \leq x]$ and the density $p^\beta(x)$ of X^β

Understand the conditional law of $Y_1, \dots, Y_n$ given $X^\beta \leq x$ ($X^\beta \geq x$)

Approximate rare-event probabilities and risk measures in the multidimensional Black-Scholes model

Design efficient Monte Carlo algorithms for precise evaluation of these quantities

Describe the behavior of stocks under stress scenarios for the index

What is to be done

Without loss of generality, it is sufficient to study:

$$X^{(m)} = \sum_{k=1}^{m} e^{Y_k} - \sum_{k=m+1}^{n} e^{Y_k}, \ m \geq 1.$$

What is to be done

Without loss of generality, it is sufficient to study:

$$X^{(m)} = \sum_{k=1}^{m} e^{Y_k} - \sum_{k=m+1}^{n} e^{Y_k}, \; m \geq 1.$$

The support of $X^{(m)}$ is

$$\begin{array}{ll} (-\infty, \infty) & \text{if} \quad 1 \leq m \leq n-1 \\ (0, \infty) & \text{if} \quad m = n \\ (-\infty, 0) & \text{if} \quad m = 0. \end{array}$$

$\Rightarrow$ we need to study

the Left tail (when $x \to 0$) of the sum $X = \sum_{k=1}^{n} e^{Y_k}$ and

the Right tail (when $x \to +\infty$) of $X^{(m)}$ for $m \geq 1$.

Related work: right tail of the sum

- Important in insurance
- For $X \geq x$ it is enough that at least one of Y_i satisfies $e^{Y_i} \geq x$.

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- For $X \geq x$ it is enough that at least one of Y_i satisfies $e^{Y_i} \geq x$.
- Asmussen and Rojas-Nandayapa (2008): the asymptotics is correlation-independent and satisfies

$$P[X > x] \sim m\overline{F}_{\mu,\sigma^2}, \quad \sigma = \max_{k=1,\dots,n} \sigma_k, \quad \mu = \max_{k:\sigma_k=\sigma} \mu_k.$$

  where $\overline{F}$ is the one-dimensional log-normal survival function and $m = \#\{k : \sigma_k = \sigma, \mu_k = \mu\}$.
- This result holds more generally for dependent subexponential random variables (Geluk and Tang, 2009).

Related work: left tail of the sum

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- For $X \leq x$ it is necessary (not sufficient) that all Y_i satisfy $e^{Y_i} \leq x$.
- Asymptotics may depend on correlation; only partial results are available in the literature. Gao et al. (2009) treat the case $n = 2$ and the case of arbitrary n under restrictive assumptions on $\mathfrak{B}$.

Outline

Notation and preliminaries

Let $\Delta_n := \{w \in \mathbb{R}^n : w_i \geq 0, i = 1, \ldots, n, \text{and } \sum_{i=1}^n w_i = 1\}\}$

and $\mathcal{E}(w) = -\sum_{i=1}^n w_i \log w_i,$ for $w \in \Delta_n$ with $0 \log 0 = 0.$

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$$\text{and}\quad \mathcal{E}(w) = -\sum_{i=1}^n w_i \log w_i, \quad \text{for } w \in \Delta_n \quad \text{with } 0 \log 0 = 0.$$

We choose $\bar{w} \in \Delta_n$ to be the unique point such that

$$\bar{w}^{\perp} \mathfrak{B} \bar{w} = \min_{w \in \Delta_n} w^{\perp} \mathfrak{B} w.$$

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With $\bar{I} := \{i \in \{1, \dots, n\} : \bar{w}_i > 0\}$ and $\bar{n} :=$ Card $\bar{I}$, assume WLOG that $\bar{I} = \{1, \dots, \bar{n}\}$.

We let $\bar{\mu} \in \mathbb{R}^{\bar{n}}$ with $\bar{\mu}_i = \mu_i$ and $\bar{\mathfrak{B}} \in M_{\bar{n}}(\mathbb{R})$ with $\bar{b}_{ij} = b_{ij}$; the elements of $\bar{\mathfrak{B}}^{-1}$ are denoted by $\bar{a}_{ij}$ and $\bar{A}_k := \sum_{j=1}^{\bar{n}} \bar{a}_{kj}$.

Assumption (A)

Our main result requires the following non-degeneracy assumption:

(A) For $i = \bar{n} + 1, \ldots, n$,

$$(e^i - \bar{w})^\perp \mathfrak{B} \bar{w} \neq 0,$$

where $e^i \in \mathbb{R}^n$ satisfies $e^i_j = 1$ if $i = j$ and $e^i_j = 0$ otherwise.

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Observe that

$$\text{grad}\, \frac{1}{2} w^\perp \mathfrak{B} w = \mathfrak{B} w.$$

Main result for X

Let Assumption (A) hold true. Then, as $x \to 0$,

$$\mathbb{P}[X \le x] = \bar{C}\left(\log\frac{1}{x}\right)^{-\frac{1+\bar{n}}{2}} \exp\left\{-\frac{(\log x - \bar{w}^{\perp}\bar{\mu} - \mathcal{E}(\bar{w}))^2}{2\bar{w}^{\perp}\mathfrak{B}\bar{w}}\right\}\left(1 + O\left(\frac{1}{|\log x|}\right)\right)$$

where

$$\bar{C} = \frac{1}{\sqrt{2\pi}\sqrt{|\bar{\mathfrak{B}}|}}\frac{\sqrt{\bar{w}^{\perp}\mathfrak{B}\bar{w}}}{\sqrt{\bar{A}_1\cdots\bar{A}_{\bar{n}}}}$$

$$\times \exp\left\{-\frac{1}{2}\sum_{i,j=1}^{\bar{n}} \bar{a}_{ij}\left(\bar{\mu}_i - \log\bar{w}_i\right)\left(\bar{\mu}_j - \log\bar{w}_j\right) + \frac{(\bar{w}^{\perp}\bar{\mu} + \mathcal{E}(\bar{w}))^2}{2\bar{w}^{\perp}\mathfrak{B}\bar{w}}\right\}.$$

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$$\times \exp \left\{ -\frac{1}{2} \sum_{i,j=1}^{\bar{n}} \bar{a}_{ij} \left(\bar{\mu}_i - \log \bar{w}_i \right) \left(\bar{\mu}_j - \log \bar{w}_j \right) + \frac{(\bar{w}^{\perp} \bar{\mu} + \mathcal{E}(\bar{w}))^2}{2 \bar{w}^{\perp} \mathfrak{B} \bar{w}} \right\}.$$

Gao et al. (2009) obtain this result under the assumption that $\bar{w}_i > 0$ for $i = 1, \ldots, n$ by applying multidimensional Laplace method.

Remarks

- $\bar{w}^{\perp}\mathfrak{B}\bar{w} \leq \min_i b_{ii}$: the left tail of the sum is, in general, thinner, than the tails of the elements.

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- For a Gaussian random variable $Y \sim N(m, \sigma^2)$,

$$\mathbb{P}[e^Y \leq x] \sim \frac{\sigma}{\sqrt{2\pi}} \left(\log \frac{1}{x}\right)^{-1} \exp\left\{-\frac{(\log x - m)^2}{2\sigma^2}\right\}$$

- The tail of a sum of log-normals may be approximated by a single log-normal variable with

$$m = \bar{w}^\perp \mu - \mathcal{E}(\bar{w}) \quad \text{and} \quad \sigma^2 = \bar{w}^\perp \mathfrak{B} \bar{w},$$

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only up to a logarithmic factor.

- The density $p(x)$ of X satisfies

$$p(x) = -\frac{\log x}{x \bar{w}^\perp \mathfrak{B} \bar{w}} \mathbb{P}[X \le x] \left(1 + O\left(\left(\log \frac{1}{x}\right)^{-1}\right)\right).$$

Example

Let $n = 2$, and $b_{11} = \sigma_1^2$, $b_{22} = \sigma_2^2$ and $b_{12} = \rho\sigma_1\sigma_2$; assume $\sigma_1 \geq \sigma_2$. Then,

$$\bar{w} = (\bar{v}, 1 - \bar{v})^\perp \quad \text{with} \quad \bar{v} = \frac{\sigma_2(\sigma_2 - \rho\sigma_1)}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \vee 0.$$

- If $\rho < \frac{\sigma_2}{\sigma_1}$, both weights are strictly positive, assumption (A) holds, and

$$p(z) \sim \frac{C}{z\sqrt{|\log z|}} e^{-\frac{1}{2}(\mu_1 + x^* - \log z, \mu_2 + y^* - \log z)\mathfrak{B}^{-1}(\mu_1 + x^* - \log z, \mu_2 + y^* - \log z)^\perp}.$$

- If $\rho > \frac{\sigma_2}{\sigma_1}$, $\bar{w} = (0, 1)^\perp$, assumption (A) holds, and

$$p(z) \sim \frac{1}{z\sigma_2\sqrt{2\pi}} e^{-\frac{(\log z - \mu_2)^2}{2\sigma_2^2}}.$$

  $\Rightarrow$ asymptotics determined by the second component.

- If $\rho = \frac{\sigma_2}{\sigma_1}$, $\bar{w} = (0, 1)^\perp$ but assumption (A) does not hold.

Outline

A special case

We first consider the case when $m = 1$, that is, we are interested in the right tail of

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The results are similar to the case of the sum: let $\mathcal{E}(w) = -\sum_{i=1}^{n} w_i \log|w_i|$,

$$\Delta_{1,n} := \{w \in \mathbb{R}^n : w_1 \geq 0, w_i \leq 0, i = 2, \ldots, n, \text{and } \sum_{i=1}^{n} w_i = 1\}$$

and choose $\bar{w} \in \Delta_{1,n}$ as the minimizer of $\min_{w \in \Delta_{1,n}} w^{\perp} \mathfrak{B} w$.
As before, we let $\bar{I} := \{i \in \{1, \ldots, n\} : \bar{w}_i \neq 0\}$ and $\bar{n} :=$ Card $\bar{I}$, and assume WLOG that $\bar{I} = \{1, \ldots, \bar{n}\}$.

Asymptotics of $X^{(1)}$

Let Assumption (A) hold true. Then, as $x \to +\infty$,

$$\mathbb{P}[X^{(1)} \geq x] = \bar{C}\,(\log x)^{-\frac{1+\bar{n}}{2}} \exp\left\{-\frac{(\log x - \bar{w}^{\perp}\bar{\mu} - \mathcal{E}(\bar{w}))^2}{2\bar{w}^{\perp}\mathfrak{B}\bar{w}}\right\}\left(1 + O\left(\frac{1}{|\log x|}\right)\right)$$

where

$$\bar{C} = \frac{1}{\sqrt{2\pi}\sqrt{|\bar{\mathfrak{B}}|}} \frac{\sqrt{\bar{w}^{\perp}\mathfrak{B}\bar{w}}}{\sqrt{|\bar{A}_1 \cdots \bar{A}_{\bar{n}}|}}$$

$$\exp\left\{-\frac{1}{2}\sum_{i,j=1}^{\bar{n}} \bar{a}_{ij}\,(\bar{\mu}_i - \log|\bar{w}_i|)\,(\bar{\mu}_j - \log|\bar{w}_j|) + \frac{(\bar{w}^{\perp}\bar{\mu} + \mathcal{E}(\bar{w}))^2}{2\bar{w}^{\perp}\mathfrak{B}\bar{w}}\right\}.$$

Comments

The exponent $\bar{w}^{\perp}\mathfrak{B}\bar{w}$ is either equal to b_{11} or less than b_{11}

- When $\bar{w}^{\perp}\mathfrak{B}\bar{w} = b_{11}$, the asymptotics is determined exclusively by Y_1
- When $\bar{w}^{\perp}\mathfrak{B}\bar{w} < b_{11}$, the asymptotics is determined by Y_1 and a subset of $(Y_2, \dots, Y_n)$.
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We call

$$v_{1,n} := \bar{w}^{\perp}\mathfrak{B}\bar{w} = \min_{w \in \Delta_{1,n}} w^{\perp}\mathfrak{B}w$$

the relative asymptotic variance of Y_1 with respect to $Y_2, \dots, Y_n$.

Asymptotics for $X^{(m)}$

Assume that for every $p = 1, \dots, m$, the covariance matrix of $Y_p, Y_{m+1}, \dots, Y_n$ satisfies Assumption (A). Then, by the previous result,

$$\mathbb{P}[e^{Y_p} - e^{Y_{m+1}} - \dots - e^{Y_n} \geq x]$$
$$= \delta_{1,p} (\log x)^{\delta_{2,p}} x^{\delta_{3,p}} \exp\left\{ -\frac{\log^2 x}{v_p} \right\} (1 + O((\log x)^{-1})), \quad p = 1, \dots, m.$$

Introduce the following parameters:

$$v = \max_{1 \leq p \leq m} v_p, \quad \mathcal{P}_4 = \{p : 1 \leq p \leq m, v_p = v\},$$
$$\delta_3 = \max_{p \in \mathcal{P}_4} \delta_{3,p}, \quad \mathcal{P}_3 = \{p \in \mathcal{P}_4 : \delta_{3,p} = \delta_3\},$$
$$\delta_2 = \max_{p \in \mathcal{P}_3} \delta_{2,p}, \quad \mathcal{P}_2 = \{p \in \mathcal{P}_3 : \delta_{2,p} = \delta_2\}.$$

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$$\mathbb{P}[X^{(m)} \geq x] = \sum_{p \in \mathcal{P}_2} \delta_{1,p} (\log x)^{\delta_2} x^{\delta_3} \exp\left\{ -\frac{1}{2} \delta_4 \log^2 x \right\} \left(1 + O\left((\log x)^{-\frac{1}{2}}\right)\right)$$

as $x \to \infty$.

- The tail behavior of $X^{(m)}$ is determined by the components of $(Y_1, \dots, Y_m)$ which have the largest relative asymptotic variance with respect to $(Y_{m+1}, \dots, Y_n)$.

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as $x \to \infty$.

- The tail behavior of $X^{(m)}$ is determined by the components of $(Y_1, \dots, Y_m)$ which have the largest relative asymptotic variance with respect to $(Y_{m+1}, \dots, Y_n)$.
- It is an extension of Theorem 1 of Asmussen – Rojas (2008), which shows that the asymptotic behavior of the right tail of $e^{Y_1} + \cdots + e^{Y_n}$ is determined by the components of $(Y_1, \dots, Y_n)$ which have the largest variance.

Outline

Using the asymptotic formulas directly

- A 4×4 covariance matrix with elements of the form $b_{ij} = \sigma_i \sigma_j \rho$ (constant correlation) with $\sigma = \{2\ 2.3\ 3\ 3\}$.
- The asymptotic approximations $F_a(x)$ and $F_a^{(2)}(x)$ of the distribution functions

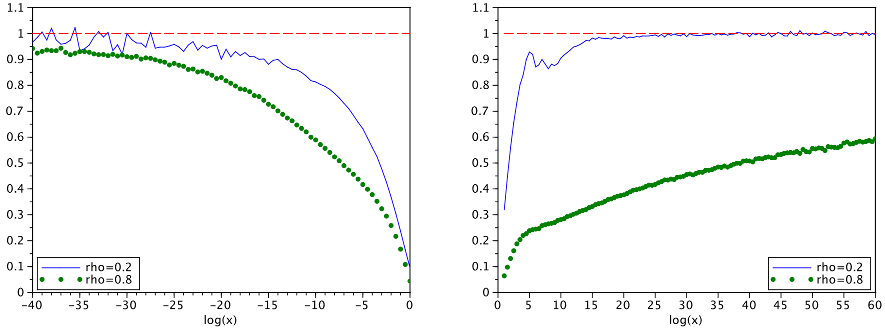
$$\mathbb{P}[X \leq x] = \mathbb{P}[e^{Y_1} + e^{Y_2} + e^{Y_3} + e^{Y_4} \leq x]$$

  and

$$\mathbb{P}[X^{(2)} \geq x] = \mathbb{P}[e^{Y_1} + e^{Y_2} - e^{Y_3} - e^{Y_4} \geq x]$$

  are compared with their Monte Carlo estimates $F_{mc}(x)$ and $F_{mc}^{(2)}(x)$.
- We plot the ratios $\frac{F_{mc}(x)}{F_a(x)}$ and $\frac{F_{mc}^{(2)}(x)}{F_a^{(2)}(x)}$ for a wide range of values of x and two values of the correlation ρ.

Using the asymptotic formulas directly



Left: $\frac{F_{mc}(x)}{F_a(x)}$. Right: $\frac{F^{(2)}_{mc}(x)}{F^{(2)}_a(x)}$. As expected, the ratios converge to one, but very slowly. On the other hand, the asymptotic formula gives the right order of magnitude for a wide range of probabilities (the values shown in the graph correspond to probabilities from 10^{-1} to 10^{-100}).

Variance reduction of Monte Carlo estimates

- The standard estimate of the distribution function $F(x) = \mathbb{P}[X \leq x]$ is

$$\widehat{F}_N(x) = \frac{1}{N}\sum_{k=1}^{N} \mathbf{1}_{\sum_{i=1}^n e^{Y_i^{(k)}} \leq x},$$

where $Y^{(1)}, \dots, Y^{(N)}$ are i.i.d. vectors with law $N(\mu, \mathfrak{B})$.

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where $Y^{(1)}, \dots, Y^{(N)}$ are i.i.d. vectors with law $N(\mu, \mathfrak{B})$.

- The variance of $\widehat{F}_N(x)$ is given by

$$\mathrm{Var}\, \widehat{F}_N(x) = \frac{F(x) - F^2(x)}{N} \sim \frac{F(x)}{N}, \quad x \to 0,$$

and the relative error

$$\frac{\sqrt{\mathrm{Var}\, \widehat{F}_N(x)}}{F(x)} \sim \frac{1}{\sqrt{NF(x)}}$$

explodes very quickly as $x \to 0$ (as $e^{c \log^2 x}$ for some constant c).

Variance reduction of Monte Carlo estimates

Rewrite the formula for F as follows:

$$F(x) = \mathbb{E}[e^{-\Lambda^\perp \mathfrak{B}^{-1}(Y-\mu)-\frac{1}{2}\Lambda^\perp \mathfrak{B}^{-1}\Lambda}\mathbf{1}_{\sum_{i=1}^n e^{Y_i+\Lambda_i}\leq x}],$$

where $\Lambda \in \mathbb{R}^n$ is a vector to be chosen such that the corresponding estimate

$$\widehat{F}_N^\Lambda(x) = \frac{1}{N}\sum_{k=1}^N e^{-\Lambda^\perp \mathfrak{B}^{-1}(Y^{(k)}-\mu)-\frac{1}{2}\Lambda^\perp \mathfrak{B}^{-1}\Lambda}\mathbf{1}_{\sum_{i=1}^n e^{Y_i^{(k)}+\Lambda_i}\leq x}$$

has variance smaller than the standard estimate.

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The variance of $\widehat{F}_N^\Lambda(x)$ is given by

$$\operatorname{Var}\widehat{F}_N^\Lambda(x) = \frac{1}{N}\left\{e^{\Lambda^\perp \mathfrak{B}^{-1}\Lambda}\mathbb{P}\left[\sum_{i=1}^n e^{Y_i-\Lambda_i}\leq x\right] - F^2(x)\right\}.$$

$\Rightarrow$ for optimal variance reduction, we need to minimize

$$V(\Lambda, x) = e^{\Lambda^\perp \mathfrak{B}^{-1}\Lambda}\mathbb{P}\left[\sum_{i=1}^n e^{Y_i-\Lambda_i}\leq x\right].$$

Variance reduction of Monte Carlo estimates

To obtain an explicit estimate, replace the probability by its asymptotic equivalent. This amounts to minimizing

$$\widetilde{V}(\Lambda, x) = \Lambda^{\perp}\mathfrak{B}^{-1}\Lambda - \frac{1}{2}\sum_{i,j=1}^{\bar{n}} \bar{a}_{ij}\left(\log(x\bar{w}_i) - \bar{\mu}_i + \Lambda_i\right)\left(\log(x\bar{w}_j) - \bar{\mu}_j + \Lambda_j\right).$$

The optimal value of Λ is given by

$$\Lambda^*_k = \sum_{i,j=1}^{\bar{n}} b_{ki}\bar{a}_{ij}\left(\log(x\bar{w}_j) - \bar{\mu}_j\right),$$

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$$\Lambda^*_k = \sum_{i,j=1}^{\bar{n}} b_{ki}\bar{a}_{ij}\left(\log(x\bar{w}_j) - \bar{\mu}_j\right),$$

and it can be shown that

$$V(\Lambda^*, x) \lesssim CF^2(x)\left(\log\frac{1}{x}\right)^{\bar{n}}$$

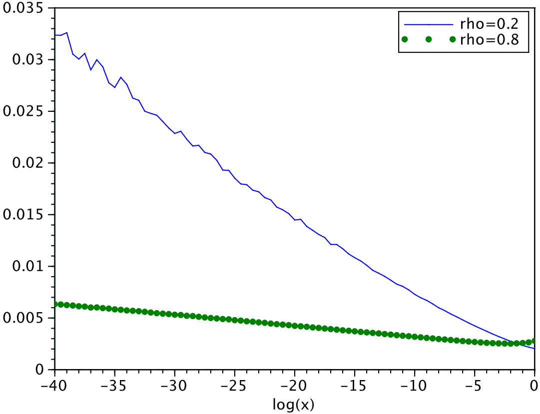
$\Rightarrow$ relative error grows only logarithmically in x as $x \to 0$.

Variance reduction of Monte Carlo estimates

$\rho = 0.2$			$\rho = 0.8$		
x	$\mathbb{P}[X \leq x]$	red. factor	x	$\mathbb{P}[X \leq x]$	red. factor
0.006738	0.0000027	152.8	0.0002035	0.0000012	269
0.01831	0.0000424	38.07	0.0009119	0.0000331	69.08
0.04979	0.0004639	14.48	0.004089	0.0005282	16.07
0.1353	0.003457	6.188	0.01832	0.005085	5.312
0.3679	0.01798	3.152	0.08209	0.02998	2.256
1.	0.06603	1.845	0.3679	0.1141	1.078

Standard deviation reduction factors obtained with the variance reduction estimate (10^6 trajectories).

Variance reduction of Monte Carlo estimates



Relative error of the variance reduction estimate.

Outline

Multidimensional Black-Scholes model

In the rest of the talk we assume that the assets $S^1, \dots, S^n$ follow a n-dimensional Black-Scholes model:

$$\log S_t = \log S_0 + bt - \frac{\mathrm{diag}(\mathfrak{B})t}{2} + \mathfrak{B}^{\frac{1}{2}} W_t,$$

where W is a standard n-dimensional Brownian motion, $\mathfrak{B}$ is a covariance matrix, $b \in \mathbb{R}$ denotes the drift vector and $\mathrm{diag}(\mathfrak{B})$ is the main diagonal of $\mathfrak{B}$.

Multidimensional Black-Scholes model

In the rest of the talk we assume that the assets $S^1, \dots, S^n$ follow a n-dimensional Black-Scholes model:

$$\log S_t = \log S_0 + bt - \frac{\mathrm{diag}(\mathfrak{B})t}{2} + \mathfrak{B}^{\frac{1}{2}} W_t,$$

where W is a standard n-dimensional Brownian motion, $\mathfrak{B}$ is a covariance matrix, $b \in \mathbb{R}$ denotes the drift vector and diag($\mathfrak{B}$) is the main diagonal of $\mathfrak{B}$. Let

$$X_t = \sum_{i=1}^{n} \xi_i S_t^i.$$

where $\xi_1, \dots, \xi_n$ are positive weights, represent a market index.

Stress testing

Assume that the fund manager holds a portfolio of assets $S^1, \ldots, S^n$ with weights $v_1, \ldots, v_n$, whose value will be denoted by

$$V_t = \sum_{i=1}^{n} v_i S_t^i.$$

In risk management, it is important to understand the behavior of the portfolio under adverse scenarios of market evolution.

Stress testing

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In risk management, it is important to understand the behavior of the portfolio under adverse scenarios of market evolution.

Assuming that the scenario corresponds to the fall of $1 - \alpha$ per cent in the index, we are interested in computing "the most likely" evolution of our portfolio which can be defined as the conditional expected value given the scenario:

$$\mathbb{E}[V_t | X_t = \alpha X_0] = \sum_{i=1}^n v_i e_i(t, \alpha X_0), \quad e_i(t, X) = \mathbb{E}[S_t^i | \sum_{k=1}^n \xi_k S_t^k = X].$$

Stress testing: main result

Let Assumption (A) hold true. Then, as $X \to 0$,

$$e_i(t,X) = X^{1+\bar{\lambda}_i} S_0^i \exp\left(b_i t - \sum_{p,q=1}^{\bar{n}} b_{pi}\bar{a}_{pq}\left(\log\frac{\bar{A}_1 + \cdots + \bar{A}_{\bar{n}}}{\bar{A}_q} + \mu_q\right)\right)$$

$$\times \exp\left(-\frac{t}{2}\sum_{p,q=1}^{\bar{n}} \bar{a}_{pq} b_{pi} b_{qi}\right)\left(1 + O\left(\left(\log\frac{1}{X}\right)^{-1}\right)\right)$$

for $i \notin \bar{I}$ and

$$e_i(t,X) = \frac{\bar{w}_i X}{\xi_i}\left(1 + O\left(\left(\log\frac{1}{X}\right)^{-1}\right)\right),$$

for $i \in \bar{I}$, where we write

$$\mu_q = \log S_0^q + \log \xi_q + b_q t - \frac{t}{2} b_{qq} \quad \text{and} \quad \bar{\lambda}_i = \frac{[\mathfrak{B}\bar{w}]_i}{\bar{w}^\perp \mathfrak{B}\bar{w}} - 1 \begin{cases} = 0, & i \in \bar{I} \\ > 0, & i \notin \bar{I} \end{cases}$$

Stress testing: remarks

The assets in the index can be classified in two categories depending on their behavior under the conditional law.

- The “safe assets” which decay proportionally to the index. These are exactly the assets which enter the Markowitz minimal variance portfolio with strictly positive weights.
- The “dangerous assets” which decay faster than the index.

Concluding remarks

- Fully explicit tail approximations for arbitrary linear combinations of log-normal random variables.
- Powerful variance reduction techniques for Monte Carlo estimation of tail events.
- Insights for financial mathematics / risk management / construction of stress scenarios.

References

- S. Asmussen and L. Rojas-Nandayapa, “Asymptotics of sums of lognormal random variables with Gaussian copula,” Statistics & Probability Letters, vol. 78, no. 16, pp. 2709–2714, 2008.
- X. Gao, H. Xu and D. Ye, “Asymptotic Behavior of Tail Density for Sum of Correlated Lognormal Variables”, International Journal of Mathematics and Mathematical Sciences, vol. 2009.
- J. Geluk and Q. Tang, “Asymptotic tail probabilities of sums of dependent subexponential random variables”, Journal of Theoretical Probability, vol. 22, no. 4, pp. 871-882, 2009.
- M. Avellaneda, D. Boyer-Olson, J. Busca and P. Friz, “Reconstructing volatility”, Risk, October 2002.
- E. Gobet and M. Miri, “Weak approximation of averaged diffusion processes”, Stoch. proc. appl., to appear.

An easy upper bound

Let $\Delta_n := \{w \in \mathbb{R}^n : w_i \geq 0, i = 1, \ldots, n, \text{and } \sum_{i=1}^n w_i = 1\}\}$

and $\mathcal{E}(w) = -\sum_{i=1}^n w_i \log w_i,$ for $w \in \Delta_n$ with $0 \log 0 = 0$.

Since $\sum_{i=1}^n e^{Y_i} = \sum_{i=1}^n w_i e^{Y_i - \log w_i} \geq \exp\left(\sum_{i=1}^n w_i Y_i + \mathcal{E}(w)\right), w \in \Delta_n,$

we conclude that

$$\mathbb{P}[X \leq x] \leq \mathbb{P}\left[\sum_{i=1}^n w_i Y_i + \mathcal{E}(w) \leq \log x\right] = N\left(\frac{\log x - \mu^\perp w - \mathcal{E}(w)}{\sqrt{w^\perp \mathfrak{B} w}}\right).$$

A reasonable bound in the tail regime is obtained by taking

$w = \arg\min_{w \in \Delta} w^\perp \mathfrak{B} w$.

$\Rightarrow$ Markowitz minimum variance portfolio!

Corollary: conditional law

Let Assumption (A) hold true. Then, as $x \to 0$, for any $u \in \mathbb{R}^n$,

$$\mathbb{E}[e^{\sum_{i=1}^n u_i(Y_i - \log x\,(1+\bar\lambda_i))}|X \leq x]$$
$$= \exp\left(\sum_{i=1}^n u_i \left\{\mu_i - \sum_{p,q=1}^{\bar n} b_{pi}\bar a_{pq}\left(\log \frac{\bar A_1 + \cdots + \bar A_{\bar n}}{\bar A_q} + \bar\mu_q\right)\right\}\right)$$
$$\times \exp\left(\frac{1}{2}\sum_{i,j\notin \bar I} u_i u_j \left\{b_{ij} - \sum_{p,q=1}^{\bar n} \bar a_{pq} b_{pi} b_{qj}\right\}\right)\left(1 + O\left(\frac{1}{|\log x|}\right)\right)$$

where $\bar\lambda_i = \frac{[\mathfrak{B}\bar w]_i}{\bar w^\perp \mathfrak{B}\bar w} - 1 \begin{cases} = 0, & i \in \bar I \\ > 0, & i \notin \bar I \end{cases}$

Same asymptotic behavior is obtained by conditioning on $\{X = x\}$

Remarks

The conditional distribution of the vector

$$Y - (1+\bar{\lambda})\log x$$

given $X \leq x$ converges weakly to the (degenerate) Gaussian law with mean

$$\mu_i' = \mu_i - \sum_{p,q=1}^{\bar{n}} b_{pi}\bar{a}_{pq}\left(\log \frac{\bar{A}_1 + \cdots + \bar{A}_{\bar{n}}}{\bar{A}_q} + \bar{\mu}_q\right)$$

and covariance matrix

$$\mathfrak{B}_{ij}' = \left\{ b_{ij} - \sum_{p,q=1}^{\bar{n}} \bar{a}_{pq} b_{pi} b_{qj} \right\} \mathbf{1}_{i,j\notin \bar{I}}.$$

Note that for $i \in \bar{I}$, the expression for μ_i' simplifies to $\mu_i' = \log \frac{\bar{A}_i}{\bar{A}_1+\cdots+\bar{A}_{\bar{n}}} = \log \bar{w}_i$.

Corollary: density

Let Assumption (A) hold true. Then, as $x \to 0$, the density $p(x)$ of X satisfies

$$p(x) = -\frac{\log x}{x\bar{w}^{\perp}\mathfrak{B}\bar{w}}\,\mathbb{P}[X \leq x]\left(1 + O\left(\left(\log\frac{1}{x}\right)^{-1}\right)\right).$$