Game-theoretic models of financial markets

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Regular random fluctuations of stock market prices are usually explained by the effect of multiple exogenous factors subjected to accidental variations.

But Brownian component in the evolution of prices on the stock market may originate from asymmetric information of stockbrokers. "Insiders" are not interested in immediate revelation of their private information. This forces them to randomize their actions and results in the appearance of the oscillatory component in price evolution.

De Meyer and Saley demonstrate this idea on a simplified model of multistage bidding between two agents for single-type risky assets.
De Meyer B., Moussa Saley H. (2002) *On the Strategic Origin of Brownian Motion in Finance.* Int. J. of Game Theory, 31, 285-319.

The MODEL

TWO PLAYERS have MONEY + SHARES of one type risky asset.
RANDOM liquidation price of a share may take TWO values
1 with probability p and 0 with $1-p$.

STEP 0: a chance move determines a liquidation price of one share ONCE FOR ALL. Both players know probability p.
Player 1 (insider) is informed on the chosen price.
Player 2 is not.
Player 2 knows that Player 1 is an insider.

STEP t, $t = 1, 2, \ldots, n$: Players propose their prices for one share,
x_t for Player 1, y_t for Player 2.
The pair (x_t, y_t) is announced to both Players.

The player who posts the LARGER price buys one share from his opponent for THIS price.

Players aim to maximize the values of their final portfolios, i.e. money plus liquidation values of obtained shares.

In this model Player 2 should use the history of Player 1's moves to update his beliefs about the liquidation price. Player 2 may re-evaluate the posterior probabilities of chance move outcome. Player 1 controls these posterior probabilities.

Thus Player 1 faces a problem of how best to use his private information without revealing it to Player 2. Player 1 must maintain a balance between taking advantage of his private information and concealing it from Player 2.

De Meyer and Saley consider the model where players may make arbitrary bids. They reduce this model to a zero-sum repeated game with lack of information on one side $G_n(p)$, as introduced by Aumann, Maschler (1966), but with continual action sets.

One-step payoffs of Player 1 for $G_n(p)$ are given by

$$a^s(x,y) = \begin{cases} y - s, & \text{if } x < y; \\ 0, & \text{if } x = y; \\ s - x, & \text{if } x > y, \end{cases}$$

where $s = 0$ or $s = 1$ is the result of chance move.

The final payoff is $\sum_{t=1}^{n} a^s(x_t, y_t)$.

The payoff function is

$$K_n(\sigma, \tau, p) = \mathbf{E}_{p,\sigma,\tau} \sum_{t=1}^{n} a^s(x_t, y_t),$$

where σ and τ are strategies of players, i.e. sequences of randomized actions at steps t, $t = 1, 2, \ldots, n$ depending on information up to the corresponding step. Besides, strategy of Player 1 depends on s but strategy of Player 2 does not.

De Meyer and Saley show that the games $G_n(p)$ have values $V_n(p)$, (i.e. the guaranteed gains of Player 1 are equal to the guaranteed losses of Player 2). They find these values

$$V_n(p)/\sqrt{n} = \int_{\chi_p}^{\infty} s f_n(s) ds,$$

where f_n is the probability density of the random variable
$S_n := \sum_{q=1}^{n} U_q/\sqrt{n}$, $U_1, \ldots, U_n$ are n independent random variables uniformly distributed over $[-1, 1]$ and χ_p is such that $p = \int_{\chi_p}^{\infty} f_n(s) ds$.

As $n \to \infty$, the TOTAL NON-AVERAGED values infinitely grow up with rate $\sqrt{n}$.

De Meyer and Saley find the optimal strategies of players.

They show that Brownian Motion in fact appears in the asymptotics of transaction prices generated by the optimal strategies.

The discrete variant of the model with bids proportional to the minimal currency unit.

The SAME informational structure.

1. The price of a share may take two integer values m and 0.

2. Any integer bids are admissible. The reasonable bids are $0, \ldots, m-1$.

The model is described by a zero-sum n-stage repeated game $G_n^m(p)$ given by the two $m \times m$ matrices of one-step payoffs of Player 1.

Thus players repeatedly play a matrix game. Player 1 knows what game is played. Player 2 knows the probability p only.

In contrast to the model with arbitrary bids, the sequence of values $V_n^m(p)$ for games $G_n^m(p)$ is bounded.

Proposition. *The value* $V_n^m(p) < H^m(p)$*, where* $H^m(p)$ *is the continuous piecewise linear function over* $[0,1]$ *with* m *domains of linearity* $[k/m,(k+1)/m]$*. For* $p = k/m$

$$H^m(k/m) = k(m-k)/2 = \mathbf{D[p]}/2.$$

The family τ^k, $k = 0, \dots, m-1$ of strategies of Player 2 ensures this upper bound. The first move τ_1^k is the action k.
The moves τ_t^k for $t > 1$ depend on the last observed pair of actions:

$$\tau_t^k(i_{t-1}, j_{t-1}) = \begin{cases} j_{t-1} - 1, & \text{if } i_{t-1} < j_{t-1}; \\ j_{t-1}, & \text{if } i_{t-1} = j_{t-1}; \\ j_{t-1} + 1, & \text{if } i_{t-1} > j_{t-1}. \end{cases}$$

So we may define correctly games $G_\infty^m(p)$ with UNLIMITED BEFOREHAND number of steps and with TOTAL NON-AVERAGED payoffs.

Solutions for games $G_\infty^m(p)$: Domansky V. (2007) *Repeated games with asymmetric information and random price fluctuations at finance markets.* Int. J. Game Theory, 2007, 36(2), 241-257.

Theorem 1. *a)* $V_\infty^m(p) = H^m(p)$.

b) For $p = k/m$, the first move of the optimal Player 1's strategy σ^k makes use of two actions $k-1$ and k. The actions occur with the total probabilities $q(k-1) = q(k) = 1/2$. The conditional posterior probabilities of the state m are

$$p(m|k-1) = (k-1)/m, \quad p(m|k) = (k+1)/m.$$

As all posterior probabilities belong to the set $p = l/m, l = 0, \dots m$, these first moves define the strategy σ^k for the games of arbitrary duration.

c) For $p \in [k/m, (k+1)/m]$ the optimal strategy of Player 2 is τ^k.

The optimal strategy of Player 1 generates a symmetric random walk of share price posterior expectations with an absorption at the extreme points 0 and m. The absorption means revealing of the true value of share by Player 2.

For the initial probability k/m, the expected duration of this symmetric random walk before absorption is $k(m-k)$.

The best response of Player 2 to the optimal strategy of Player 1 provides him the fixed loss of 1/2 per step.

The value of the infinite game $V_{\infty}^{m}(k/m) = k(m-k)/2$ is equal to the expected number of steps before its termination multiplied by Player 1's gain per step 1/2.

Sandomirskaia M. (2013) On the Bidding with Asymmetric Information and the Finite Number of Repeatition. Int. Conf. Game Theory and Management GTM2013. Abstracts. 203-206.

Theorem 2. *If Player 1 exploits the strategy σ^k in the game $G_n^m(k/m)$, then his guaranteed gain is*
$\min_\tau K_n^m(\sigma^k, \tau, k/m) = \frac{(m-k)k}{2} - \varepsilon_n^m(k)$, *where*

$$\varepsilon_n^m(k) = \frac{1}{2m} \sum_{l=1}^{[m/2]} \cos^n \frac{\pi(2l-1)}{m} \sin \frac{\pi k(2l-1)}{m} \cdot \frac{\cos \frac{\pi(2l-1)}{2m}}{\sin^3 \frac{\pi(2l-1)}{2m}},$$

with $[\alpha]$ being the integer part of α.
Corollary. $\varepsilon_n^m(k) = O(\cos^n(\pi/m))$, *i.e. it decreases exponentially. As*

$$V_\infty^m(k/m) = \frac{(m-k)k}{2} > V_n^m(k/m) > \min_\tau K_n^m(\sigma^k, \tau, k/m) = \frac{(m-k)k}{2} - \varepsilon_n^m(k),$$

the strategy σ^k is a ε-optimal strategy of Player 1 for the game $G_n^m(p)$, where $\varepsilon < \varepsilon_n^m(k)$.

Bidding games with countable state and action spaces

Domansky V., Kreps V. (2009) *Repeated games with asymmetric information and random price fluctuations at finance markets: the case of countable state space.* Centre d'Economie de la Sorbonne. Univ. Paris 1. Pantheon – Sorbonne. Preprint 2009.40. http://ces.univ-paris1.fr/cesdp/CESFramDP2009.htm

We consider the same model, but RANDOM liquidation price of a share may take any INTEGER value according to a probability distribution $\mathbf{p}$ over $\mathbb{Z}^1$.
This n-stage model is described by a repeated game $G_n(\mathbf{p})$ with incomplete information of Player 2 with countable state and action spaces $S = I = J = \mathbb{Z}^1$.

If the share price has a finite variance $\mathbf{D}[\mathbf{p}]$, then the sequence $V_n(\mathbf{p})$ is bounded and we solve the game $G_\infty(\mathbf{p})$ of unlimited duration.

We construct the optimal strategies of Player 1 as combinations of his optimal strategies for "elementary" games with two states.

To do this we use the symmetric representation of probability distributions $\mathbf{p} \in \Theta(r) = \{\mathbf{p} : \mathbf{E}[\mathbf{p}] = r\}$ over the integer lattice $\mathbb{Z}^1$ with a fixed integer mean value r as probability mixtures of distributions with not more than two-point supports with the same mean value.

We give such representation for centered distributions $\mathbf{p} \in \Theta(0)$:

$$\mathbf{p} = \mathbf{p}(0) \cdot \delta^0 + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{k+l}{\sum_{t=1}^{\infty} t \cdot \mathbf{p}(t)} \mathbf{p}(-l)\mathbf{p}(k) \cdot \mathbf{p}^0_{k,-l}, \qquad (1)$$

where δ^0 is the degenerate distribution with one-point support $\{0\}$; $\mathbf{p}^0_{k,-l}$ is the centered distributions with two-point supports $\{-l, k\}$.

Optimal strategies $\sigma^0(\mathbf{p})$ of Pl.1 for games $G_\infty(\mathbf{p})$ with $\mathbf{D}[\mathbf{p}] < \infty$ and $\mathbf{E}[\mathbf{p}] = 0$

a) If a chance move chooses 0, then Player 1 stops the game.
b) If a chance move chooses $z = k$ or $z = -l$, where $k, l \in \mathbb{N}$, then Player 1 chooses a point $z_2 = -l$ or $z = k$ by means of lottery with probabilities

$$\mathbf{P}_{\mathbf{p}}(\mathbf{p}^0_{k,-l}|k) = \frac{l \cdot \mathbf{p}(-l)}{\sum_{t=1}^{\infty} t \cdot \mathbf{p}(t)}; \quad \mathbf{P}_{\mathbf{p}}(\mathbf{p}^0_{k,-l}|-l) = \frac{k \cdot \mathbf{p}(k)}{\sum_{t=1}^{\infty} t \cdot \mathbf{p}(t)},$$

that are conditional probabilities of two-point distributions given one point in their supports, corresponding to probability mixture (1).
c) Player 1 plays the optimal strategy $\sigma^0(\cdot|z)$ for the state z in the two-point game $G_\infty(\mathbf{p}^0_{k,-l})$.

This strategy guarantees Player 1 the gain

$$V_\infty(\mathbf{p}) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{k+l}{\sum_{t=1}^{\infty} t \cdot \mathbf{p}(t)} \mathbf{p}(-l)\mathbf{p}(k) \cdot \mathbf{D}[\mathbf{p}^0_{k,-l}]/2 = \mathbf{D}[\mathbf{p}]/2,$$

as the variance $\mathbf{D}[\mathbf{p}]$ is a linear function over $\Theta(0)$.

Theorem 3. *For distributions $\mathbf{p}$ with finite variances $\mathbf{D}[\mathbf{p}]$, the values $V_\infty(\mathbf{p})$ are given by a continuous, concave, and piecewise linear function H. Its domains of linearity are*

$$\{\mathbf{p} : \mathbf{E}[\mathbf{p}] \in [r, r+1]\}, \quad r \in \mathbb{Z}.$$

The function H is defined by its values

$$H(\mathbf{p}) = \mathbf{D}[\mathbf{p}]/2,$$

at break hyperplanes

$$\Theta(r) = \{\mathbf{p} : \mathbf{E}[\mathbf{p}] = r\}.$$

For $\mathbf{p} : \mathbf{E}[\mathbf{p}] \in [r, r+1]$ $\quad \tau^r$ is the optimal strategy of Player 2.

For $\mathbf{p} \in \Theta(r)$ $\quad \sigma^r$ is the optimal strategy of Player 1.

The strategy σ^r of Player 1 generates a symmetric random walk of share price posterior expectations with an absorption that may occur at any stage. The absorption means revealing of the true value of share by Player 2.

Multistage bidding with risky assets of two types.

Victor Domansky, Victoria Kreps (2013) *Repeated games with asymmetric information modeling financial markets with two risky assets*. RAIRO-Oper. Res. Vol.47, Is.3, 251–272.

Victor Domansky (2013) *Symmetric representations of bivariate distributions*. Statistics and Probability Letters, **83**, 1054–1061.

General trading mechanism

Bernard De Meyer (2010) *Price dynamics on a stock market with asymmetric information.* Games and Economic Behavior, **69(1)**, 42-71.