A Lévy HJM multiple-curve model with application to CVA computation

Stéphane Crépey
Université d'Evry, Laboratoire Analyse & Probabilités

Advanced Methods in Mathematical Finance
Université d'Angers, 2-7 September 2013

This research benefited from the support of the "Chair Markets in Transition" under the aegis of Louis Bachelier laboratory, a joint initiative of École polytechnique, Université d'Évry Val d'Essonne and Fédération Bancaire Française

Focus of this talk

Illustration of the interaction between the multiple-curve and the counterparty risk/funding issues, in the case of interest-rate derivatives

- Clean valuation = derivation of a "fully collateralized" price P_t at an OIS collateral rate
  - Fully collateralized at an OIS collateral rate→ no CVA/DVA/LVA/RC
  - OIS discounting versus Libor fixings → Multiple-curve
- Computation of a CVA+DVA+LVA+RC=TVA correction Θ_t to account for counterparty risk and excess-funding costs
  - Θ_0 = price of a dividend-paying option on P_τ

τ	(first) default time of a party
Dividends	Excess-funding benefit/cost

Outline

Libor

Most interest-rate derivatives have Libor-indexed cash-flows (Libor fixings)

What is Libor?

- Libor stands for London InterBank Offered Rate. It is produced for 10 currencies with 15 maturities quoted for each, ranging from overnight to 12 Months producing 150 rates each business day. Libor is computed as a trimmed average of the interbank borrowing rates assembled from the Libor contributing banks.
- More precisely, every contributing bank has to submit an answer to the following question: "At what rate could you borrow funds, were you to do so by asking for and then accepting inter-bank offers in a reasonable market size just prior to 11 am?"

OIS

In most currencies there is also an interbank market of overnight loans, at a rate dubbed OIS (spot) rate in reference to the related swap market

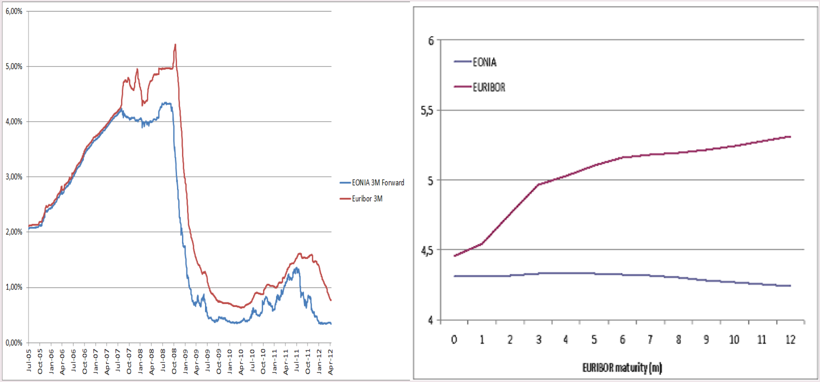
- In some currencies the OIS rate (like the Eonia rate for the euro) can be viewed as a short-tenor limit of Libor
- In others (like US dollar) this view is simplistic since the panel of the Libor and of the OIS rate is not the same, and the OIS rate reflects actual transaction rates (as opposed to a purely collected Libor)

LOIS

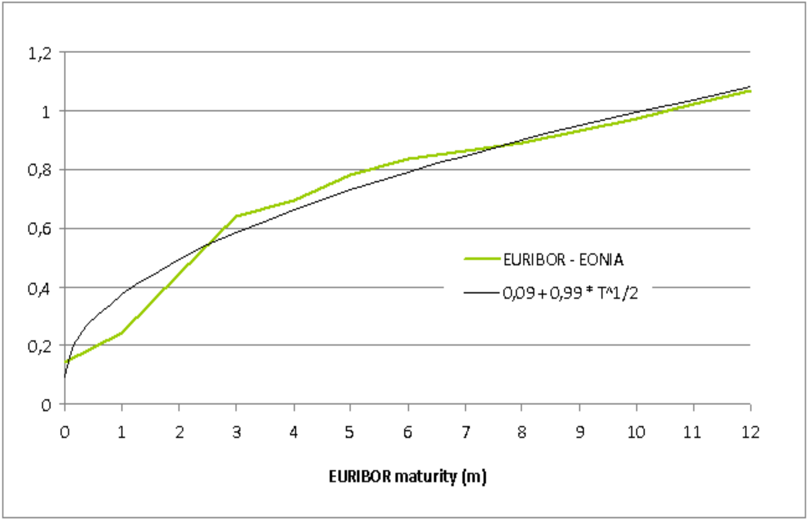
Divergence Euribor (“L”) / Eonia-swap (“R”) rates

Left: Sudden divergence between the 3m Euribor and the 3m Eonia-swap rate that occurred on Aug 6 2007

Right: Term structure of Euribor vs Eonia-swap rates, Aug 14 2008



Square root fit of the LOIS corresponding to the data of Aug 14 2008



→ Talk by Raphaël DOUADY on Thursday

Outline

Multiple-Curve Clean Valuation of Interest Rate Derivatives

Generic clean valuation formula $\beta_t P_t = \mathbb{E}\left(\int_t^T \beta_s dD_s \,\middle|\, \mathcal{F}_t\right)$ with $\beta_t = e^{-\int_0^t r_s ds}$

- Case of a single payoff χ at time T:

$$P_t = B_t^T \mathbb{E}^T[\chi|\mathcal{F}_t]$$

  - T-forward neutral measure $\mathbb{Q}^T$ and T-ZC price process B^T
- Libor fixings dD_t
- Appropriate choice of the OIS rate as the clean discount rate r_t
  - Perverse incentives for traders otherwise
  - Calibration constraints to market data = clean prices discounted at OIS

OIS discounting versus Libor fixings

- In a multiple curve environment one loses the usual consistency between discounting and fixing of classical one-curve interest rates models

→ Increased complexity of clean valuation of Libor derivatives

Markovian perspective

- With TVA in mind, "static" calibrability is not the only clean valuation tractability issue
- TVA $\sim$ option on P_τ $\rightarrow$ Tractability should also be considered at the dynamic level of plugging a clean price process P_t into a TVA Monte-Carlo or American Monte Carlo (numerical BSDE) engine
  - American Monte Carlo valuation of the TVA Θ_t and sometimes even of its "underlying" P_t

    Cesari, G. et al.: *Modelling, Pricing, and Hedging Counterparty Credit Exposure*. Springer Finance, 2010.
  - Or marked branching diffusion approach (particles)

    Henry-Labordère, P.: Cutting CVA's complexity, *Risk* 2012.

$\rightarrow$ Markovian perspective on a clean price process P_t also key

Affine diffusions versus Lévy drivers

- Tractable calibration possible either with affine diffusions or by means of (possibly time-inhomogenous) Lévy drivers
- Less factors should be the focus with TVA in mind

Interest rate derivatives

- Forward rate agreement (FRA) for the future time interval $[T, T+\delta]$:
  payoff at time $T+\delta$ is (for physical delivery) $\delta\left(L_{T,T+\delta} - K\right)$
- Floating-for-fixed interest rate swap with maturity T_n, the payment dates $T_1 < \ldots < T_n$ and the fixed rate S, and starting at $T_0 \geq 0$:
  cash flows at each payment date are $\delta_{k-1}\left(L_{T_{k-1},T_k} - S\right)$
- Caplet with strike K and maturity T_k, settled in arrears:
  payoff at settlement date is T_{k+1} $\left(L_{T_k,T_{k+1}} - K\right)^+$
- Swaption with swap rate S and exercise date T – option to enter an interest rate swap:
  payoff at exercise date is $(P_T^{sw}(T_1, T_n, S))^+$, i.e. positive part of the value of the underlying swap at T

Interest rate derivatives – WRONG since the crisis

- A Libor can be seen as risk-free
- A FRA rate

$$L_t^{T,S} = \mathbb{E}_t^S L_{T,S}$$

(that makes a FRA value equal to zero at time t) is given in terms of risk-free zero coupon bond prices as

$$\frac{1}{\delta}\left(\frac{B_t^T}{B_t^{T+\delta}} - 1\right)$$

- The value of a Libor interest rate swap at time t is simply $1 - B_t^{T_n} - S\sum_{k=1}^{n} \delta_{k-1} B_t^{T_k}$
- A caplet can be transformed into a put option on a zero coupon bond
- A swaption can be seen as a put option on a coupon bearing bond

Multiple Curve (Clean Valuation) Models

- Short-rate model of Kenyon (2010)
- Market models of Mercurio (2009, 2010)
  - Setting a new market standard in terms of the FRA rates $L_t^{T,S}$
- Market model of Bianchetti (2009)
  - Cross-currency mathematical framework
- HJM multi-currency model of Fujii et al. (2010)
  - Choice of collateral currency and embedded cheapest-to-deliver option
- Hybrid HJM-Market "parsimonious" models of Moreni and Pallavicini (2010 and 2012)
  - The best of both worlds?
- Affine (short-rate) interbank risk model of Filipović and Trolle (2011), defaultable Lévy HJM model of Crépey al. (2011)...

Outline

Lévy HJM Multiple-Curve Model

S. Crépey, Z. Grbac, N. Ngor and D. Skovmand: A Lévy HJM multiple-curve model with application to CVA computation *(in preparation)*.

- We model directly observed (or bootstrapped) quantities: B_t^T and $(S-T)\times$ the FRA rates, i.e.

$$F_t^{T,S} = (S-T)\mathbb{E}_t^S L_{T,S} \tag{1}$$

  - With TVA computations later in mind, we do it in a HJM and Lévy-driven setup
- Low-dimensional Lévy-driven Markovian short-term specifications X_t
  - $P_t = P(t, X_t)$ for all vanilla interest rate derivatives as required for CVA computations
  - Vector factor process X_t made of
    - An OIS short rate process r_t
    - An auxiliary process q_t explaining the divergence between the Libor markets of different tenor
    - Auxiliary processes if need be for the Markov sake

- The driving process is a d-variate Lévy process Y with cumulant $\mathbb{E}[e^{zY_1}] = \psi(z)$.
- The OIS bond price is given in a classical HJM fashion as

$$B_t^T = \frac{B_0^T}{B_0^t} \exp\left(\int_0^t (A^t(s) - A^T(s))ds + \int_0^t (\Sigma^t(s) - \Sigma^T(s))dY_s \right)$$

where A and Σ are deterministic, real-valued functions satisfying the drift condition $A^T(s) = \psi(-\Sigma^T(s))$.

- We model $F^{T,S}$ as

$$F_t^{T,S} = F_0^{T,S} \exp\left(\int_0^t \alpha^{T,S}(s)ds + \int_0^t \varsigma^{T,S}(s)dY_s \right) \qquad (2)$$

where α is a drift, ς a volatility and Δ a positive shift.

Lévy Hull-White Specification

- $Y = (Y^1, Y^2)$ made of two independent NIG components, with respective cumulants of the form

$$\psi_i(z) = -\nu_i \left(\sqrt{\nu_i^2 - 2z_i\theta_i - z_i^2\sigma_i^2} - \nu_i \right), \quad i = 1, 2, \tag{3}$$

where $\nu_i, \sigma_i > 0$ and $\theta_i \in \mathbb{R}$, for $i = 1, 2$.

- Volatility structures of Vasicek type. More precisely, volatility of B_t^T

$$\Sigma^T(s) = \left(\frac{\sigma}{a} \left(1 - e^{-a(T-s)} \right), 0 \right)$$

and volatility of $F^{T,S}$

$$\varsigma^{T,S}(s) = \left(\frac{\sigma}{a} e^{as} \left(e^{-aT} - e^{-aS} \right), \frac{\sigma^*(T,S)}{a^*} e^{a^*s} \left(e^{-a^*T} - e^{-a^*S} \right) \right), \tag{4}$$

where $\sigma, \sigma^*(T,S) > 0$ and $a, a^* \neq 0$ are real constants.

→ OIS bond price of the form

$$B_t^T = \exp(m^T(t) + n^T(t) r_t), \tag{5}$$

where the dynamics of the short rate r is

$$dr_t = a(\rho(t) - r_t)dt + \sigma dY_t^1$$

→ FRA Rate

$$\begin{aligned} F_t^{T,S} &= F_0^{T,S} \exp\left(\int_0^t \alpha^{T,S}(s)ds + \frac{\sigma}{a}\left(e^{-aT} - e^{-aS}\right) \int_0^t e^{as} dY_s^1 \right. \\ &\quad \left. + \frac{\sigma^*(T,S)}{a^*}\left(e^{-a^*T} - e^{-a^*S}\right) \int_0^t e^{a^*s} dY_s^2 \right) \\ &= \exp\left(m^{T,S}(t) + n^{T,S}(t) r_t + p^{T,S}(t) q_t\right), \end{aligned} \tag{6}$$

where

$$dq_t = -a^* q_t dt + dY_t^2, \qquad q_0 = 0.$$

Basis swap

- A basis swap is an interest rate swap, where two floating payments linked to the Libors with different tenors are exchanged (e.g. 3m vs. 6m-Libor or 6m vs. 12m-Libor).
- Two tenor structures: $\mathcal{T}^1 = \{T_0^1 < \ldots < T_{n_1}^1\}$ and $\mathcal{T}^1 = \{T_0^2 < \ldots < T_{n_2}^2\}$, where $T_0^1 = T_0^2 > 0$, $T_{n_1}^1 = T_{n_2}^2$, and $\mathcal{T}^1 \subset \mathcal{T}^2$.
- The swap is initiated at time T_0^1, where the first payments are due at T_1^1 and T_1^2. For $t \leq T_0^1$, the time-t value is given by

$$P_t^{bsw} = N \left(\sum_{i=1}^{n_1} B_t^{T_i^1} F_t^{T_{i-1}^1, T_i^1} - \sum_{j=1}^{n_2} B_t^{T_j^2} F_t^{T_{j-1}^2, T_j^2} \right).$$

- Note that in general $P_t^{bsw} \neq 0 \Rightarrow$ basis swap spread $S^{bsw} > 0$ which is added to the smaller tenor leg

- In the pre-crisis one-curve setting the value of such a basis swap was zero since

$$\begin{aligned} P_t^{bsw} &= (B_t^{T_0^1} - B_t^{T_{n_1}^1}) - (B_t^{T_0^2} - B_t^{T_{n_2}^1}) \\ &= 0. \end{aligned}$$

- In the multiple-curve setup we cannot use this “telescopic” simplification since the FRA rates got disconnected from the Bs.

Caplet

Consider a caplet with strike K and maturity T, settled in arrears

$$\begin{aligned} P^{Cpl}(0; T, K) &= \delta\, B_0^{T+\delta}\mathbb{E}^{T+\delta}\left[(L_{T,T+\delta} - K)^+\right] \\ &= B_0^{T+\delta}\mathbb{E}^{T+\delta}\left[\left(F_T^{T,T+\delta} - \delta K\right)^+\right] \\ &= B_0^{T+\delta}\mathbb{E}^{T+\delta}\left[\left(e^X - \bar{K}\right)^+\right] \\ &= \frac{B_0^{T+\delta}}{2\pi}\int_{\mathbb{R}} \frac{\bar{K}^{1+\mathrm{i}v-R} M_X^{T+\delta}(R-\mathrm{i}v)}{(\mathrm{i}v-R)(1+\mathrm{i}v-R)}dv \end{aligned}$$

where $\bar{K} = \delta K + \Delta_{T,S}$ and for any $R \in (1, +\infty)$ such that $M_X^{T+\delta}$ is well-defined:

$$M_X^{T+\delta}(z) = \mathbb{E}^{T+\delta}\left[e^{zX}\right],$$

for random variable

$$X = \log(F_0^{T,T+\delta} + \Delta_{T,S}) + \int_0^T \alpha(s, T, T+\delta)ds + \int_0^T \varsigma(s, T, T+\delta)dY_s$$

Swaption

Consider a swaption on the swap defined earlier. Its value at time $t = 0$ is given by

$$
\begin{aligned}
P_0^{swn} &= B_0^T \mathbb{E}^T \left[\sum_{j=1}^{n} \delta_{j-1} B_T^{T_j} \left(S(T; T_1, T_n) - S \right)^+ \right] \\
&= B_0^T \mathbb{E}^T \left[\left(\sum_{j=1}^{n} B_T^{T_j} F_T^{T_{j-1}, T_j} - \sum_{j=1}^{n} \delta_{j-1} B_T^{T_j} S \right)^+ \right] \\
&= B_0^T \mathbb{E}^T \left[\left(\sum_{j=1}^{n} a^{j,0} e^{a^{j,1} X_T^1 + a^{j,2} X_T^2} - \sum_{j=1}^{n} b^{j,0} e^{b^{j,1} X_T^1} \right)^+ \right] \\
&= B_0^T \mathbb{E}^T f(X_T^1, X_T^2),
\end{aligned}
$$

where

$$
X_T = \left(\int_0^T e^{as} dY_s^1, \int_0^T e^{a^* s} dY_s^2 \right).
$$

Swaption

- Time-0 price of a swaption with swap rate S and maturity T_n:

$$P_0^{swn} = \frac{B_0^T}{(2\pi)^2} \int_{\mathbb{R}^2} M_{X_T}^T(R + \mathrm{i}\, u)\widehat{f}(\mathrm{i}\, R - u)du,$$

where $\widehat{f}$ is the bivariate Fourier transform of f and $R \in \mathbb{R}^2$ is such that $M_{Y_T}^T(R)$ given below exists

$$\begin{aligned} M_{X_T}^T(z) &= \mathbb{E}^T\left[e^{z_1 X_T^1 + z_2 X_T^2}\right] = \mathbb{E}^T\left[e^{\int_0^T z_1 e^{as} dY_s^1 + \int_0^T z_2 e^{a^* s} dY_s^2}\right] \\ &= \exp\left(-\int_0^T \psi\left(\frac{\sigma}{a}(1 - e^{-a(T-s)}), 0\right) ds\right) \\ &\quad \times \exp\left(\int_0^T \psi\left(z_1 e^{as} - \frac{\sigma}{a}(1 - e^{-a(T-s)}), z_2 e^{a^* s}\right) ds\right) \end{aligned}$$

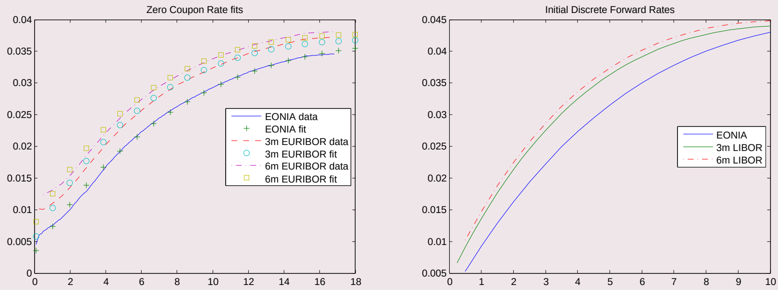
- Linear boundary approximation method of Singleton and Umantsev (2002).

Outline

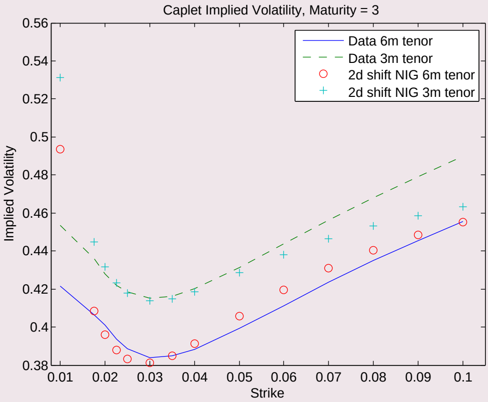
Calibration Results

- Bloomberg EUR market data from the 4th of January 2011
- Regularized initial Eonia, 3m-Euribor and 6m-Euribor term structures fitted to Nelson-Siegel-Svensson parameterizations

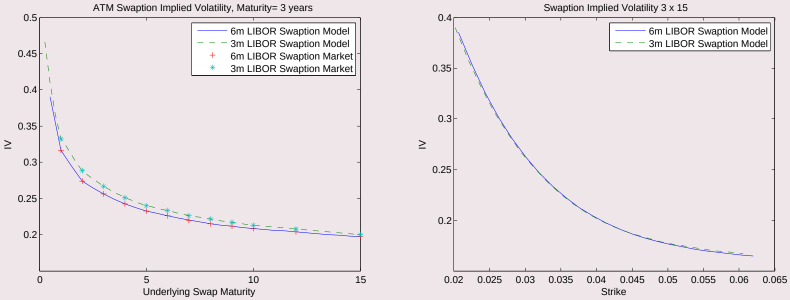
Initial term structures. *Left:* Zero coupon rates. *Right:* Discrete forward rates.



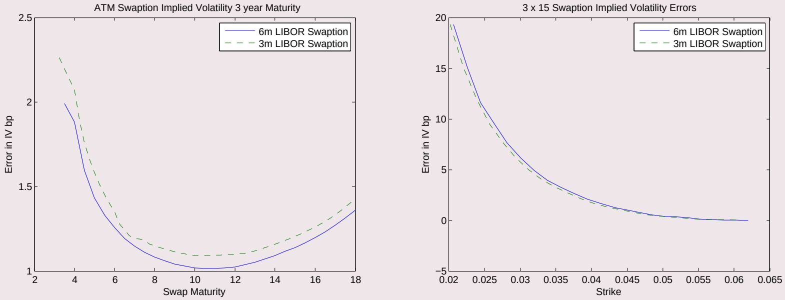
Market versus calibrated caplet implied volatilities.



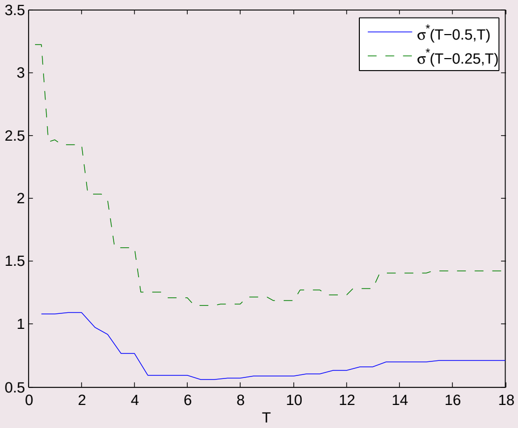
Model swaption implied volatility.
Left: implied volatility of at-the-money swaptions with maturity of 3 years and varying swap lengths is compared to market data. 3m and 6m refers to payment frequency and tenor of the floating LIBOR rate of the underlying swap.
Right: implied volatility of a 3y$\times$15y swaption with varying strikes



Errors due to the linear boundary approximation calculated for swaptions with calibrated parameters.
Left: error implied volatility in basis points of at-the-money swaptions with varying swap lengths. The error is calculated as $10^4\times$ (5M-MC impld vol - approx impld vol).
Right: error implied volatility error of a 3y$\times$15y swaption with varying strikes.



Calibrated values of $\sigma^*(T,S)$



TVA computations

- Funding spread coefficient

$$g_t(\pi, \varsigma) = g_t(\pi) = b_t\Gamma_t^+ - \bar{b}_t\Gamma_t^- + c_t\big(\pi - \Gamma_t\big)^+ - \bar{c}_t\big(\pi - \Gamma_t\big)^-$$

b and $\bar{b}$ bases over the risk-free rate for the remuneration of the collateral Γ

c and $\bar{c}$ bases corresponding to the remuneration of the external funding debt of the bank

→ Pre-default TVA equation written as the following BSDE, under the risk-neutral neutral measure $\mathbb{P}$:

$$\Theta_t = \mathbb{E}_t \int_t^T \widetilde{g}(P_s - \Theta_s)ds\,, \quad t \in [0, T]$$

$$\widetilde{g}_t(P_t - \vartheta)) + r_t\vartheta = - \underbrace{\gamma_t \bar{p}_t (1-\bar{\rho})(Q_t - \Gamma_t)^-}_{\text{costly Crebit Valuation Adjustment (CVA)}}$$

$$+ \underbrace{\gamma_t p_t ((1-\rho)(Q_t - \Gamma_t)^+}_{\text{beneficial Debit Valuation Adjustment (DVA)}}$$

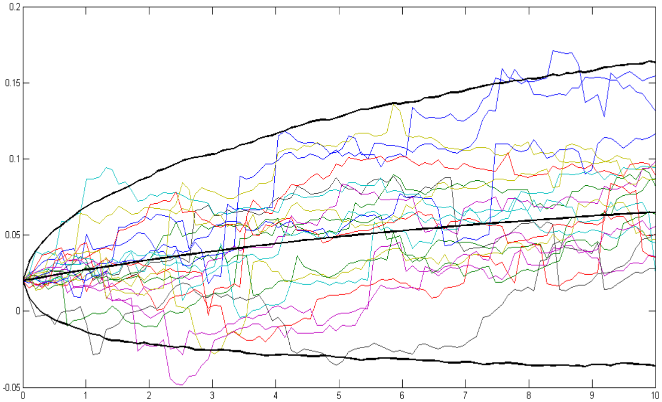
$$+ \underbrace{b_t \Gamma_t^+ - \bar{b}_t \Gamma_t^- + c_t (P_t - \vartheta - \Gamma_t)^+ - \widetilde{c}_t (P_t - \vartheta - \Gamma_t)^-}_{\text{excess-funding benefit/cost Liquidity Valuation Adjustment(LVA)}}$$

$$+ \underbrace{\gamma_t (P_t - \vartheta - Q_t)}_{\text{Replacement Cost (RC)}}$$

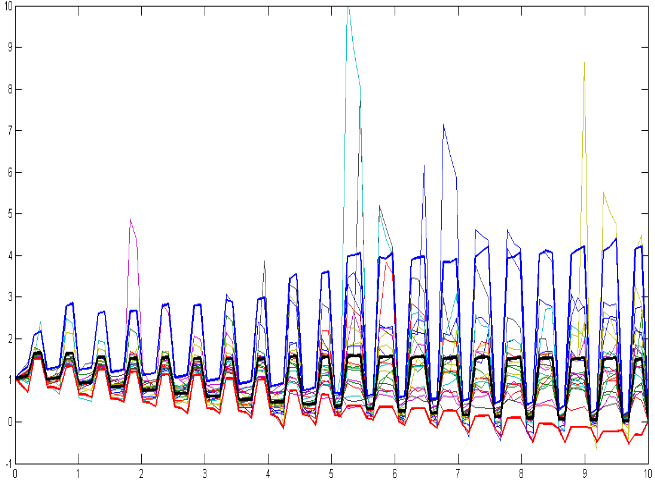
- $\widetilde{c}_t := \bar{c}_t - \gamma_t p_t (1 - \mathfrak{r})$ External borrowing basis net of the credit spread
  - Liquidity borrowing funding basis
- The positive (negative) TVA terms can be considered as "deal facilitating" ("deal hindering") as they decrease the TVA and therefore increase the price (cost of the hedge) for the bank

Each ensuing panel shows twenty paths of a process simulated with $n = 100$ time points, along with the process mean and 2.5 / 97.5-percentiles computed as function of time over $m = 10^4$ simulated paths.

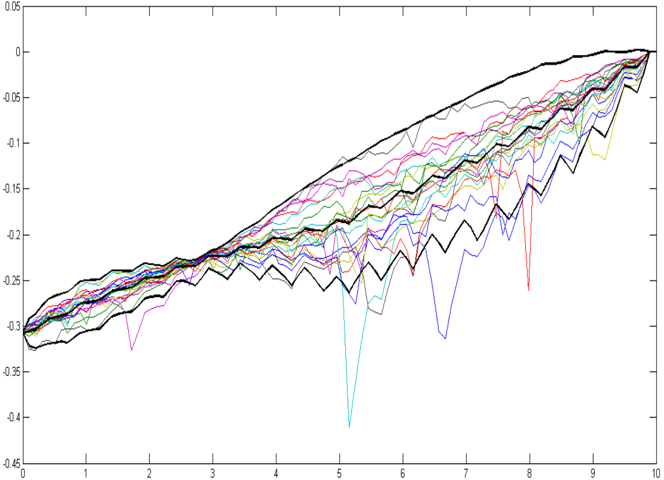
Calibrated short rate process r_t



Basis swap clean value process $P_t = P(t, X_t)$ where $X_t = (r_t, q_t, r_t^1, q_t^1, r_t^2, q_t^2)$



Basis swap TVA process $\Theta_t = \Theta(t, X_t)$



Basis swap CVA/DVA/LVA/RC Expected Exposures