Semi-Markov model for market microstructure and HF trading

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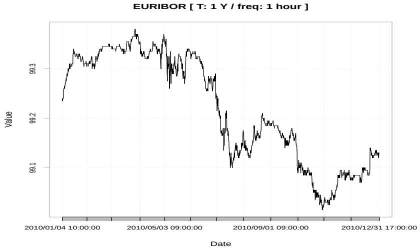
Financial data modelling

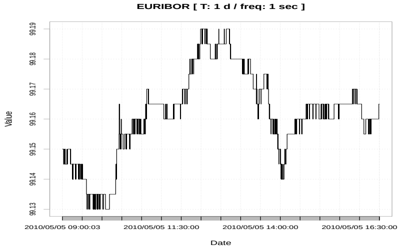
- Continuous time price process $(P_t)_t$ over $[0, T]$ observed at

$$P_0, P_\tau, \ldots, P_{n\tau}$$

- Different modelling of P according to scales τ and T:
  - Macroscopic scale (hourly, daily observation data): Itô semimartingale
  - Microscopic scale (tick data) $\rightarrow$ High frequency

Euribor contract, 2010, for different observation scales





Stylized facts on HF data

Microstructure effects

- Discrete prices: tick data
- Irregular spacing of jump times: clustering of trading activity
- Mean-reversion: negative autocorrelation of consecutive variation prices

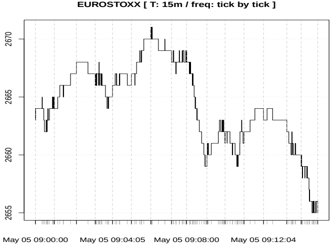


Figure: Eurostoxx contract, 2010 may 5, 9h-9h15, tick frequency

Limit order book

- Most of modern equities exchanges organized through a mechanism of *Limit Order Book* (LOB):

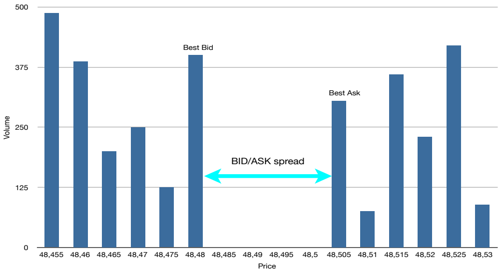


Figure: Instantaneous picture of a LOB

High frequency finance

Two main streams in literature:

- Models of intra-day asset price
  - Latent process approach: Gloter and Jacod (01), Ait Sahalia, Mykland and Zhang (05), Robert and Rosenbaum (11), etc
  - Point process approach: Bauwens and Hautsch (06), Cont and de Larrard (10), Bacry et al. (11), Abergel, Jedidi (11),

$\rightarrow$ Sophisticated models intended to reproduce microstructure effects, often for purpose of volatility estimation

- High frequency trading problems
  - Liquidation and market making in a LOB: Almgren, Cris (03), Alfonsi and Schied (10, 11), Avellaneda and Stoikov (08), etc

$\rightarrow$ Stochastic control techniques for optimal trading strategies based on classical models of asset price (arithmetic or geometric Brownian motion, diffusion models)

Objective

- Make a "bridge" between these two streams of literature:

▶ Construct a "simple " model for asset price in Limit Order Book (LOB)

- **realistic**: captures main stylized facts of microstructure
  - Diffuses on a macroscopic scale
  - Easy to estimate and simulate
- **tractable** (simple to analyze and implement) for dynamic optimization problem in high frequency trading

$\rightarrow$ Markov renewal and semi-Markov model approach

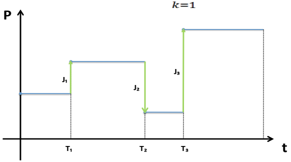
Model-free description of asset mid-price (constant bid-ask spread)

Marked point process

Evolution of the univariate mid price process (P_t) determined by:

- The **timestamps** $(T_k)_k$ of its jump times $\leftrightarrow$ N_t counting process: $N_t = \inf\{n : \sum_{k=1}^{n} T_k \leq t\}$: modeling of volatility clustering, i.e. presence of spikes in intensity of market activity
- The **marks** $(J_k)_k$ valued in $\mathbb{Z} \setminus \{0\}$, representing (modulo the tick size) the price increment at T_k: modeling of the microstructure noise via mean-reversion of price increments

$$P_t = P_0 + 2\delta \sum_{k=1}^{N_t} J_k$$



Semi-Markov model approach

Markov Renewal Process (MRP) to describe $(T_k, J_k)_k$.

• Largely used in reliability

• Independent paper by d'Amico and Petroni (13) using also semi Markov model for asset prices

Jump side modeling

For simplicity, we assume $|J_k| = 1$ (on data, this is true 99,9% of the times) :

- J_k valued in $\{+1, -1\}$: side of the jump (upwards or downwards)

$$J_k = J_{k-1} B_k \tag{1}$$

$(B_k)_k$ i.i.d. with law: $\mathbb{P}[B_k = \pm 1] = \frac{1 \pm \alpha}{2}$ with $\alpha \in [-1, 1)$.
$\leftrightarrow$ $(J_k)_k$ irreducible Markov chain with symmetric transition matrix:

$$Q_\alpha = \begin{pmatrix} \frac{1+\alpha}{2} & \frac{1-\alpha}{2} \\ \frac{1-\alpha}{2} & \frac{1+\alpha}{2} \end{pmatrix}$$

Remark: arbitrary random jump size can be easily considered by introducing an i.i.d. multiplication factor in (1).

Mean reversion

- Under the stationary probability of $(J_k)_k$, we have:

$$\alpha = \text{correlation}(J_k, J_{k-1})$$

- Estimation of α:

$$\hat{\alpha}_n = \frac{1}{n}\sum_{k=1}^{n} J_k J_{k-1}$$

$\rightarrow \alpha \simeq -87,5\%$, (Euribor3m, 2010, 10h-14h)
$\rightarrow$ Strong mean reversion of price returns

Timestamp modeling

Conditionally on $\{J_k J_{k-1} = \pm 1\}$, the sequence of inter-arrival jump times $\{S_k = T_k - T_{k-1}\}$ is i.i.d. with distribution function $F_\pm$ and density $f_\pm$:

$$F_\pm(t) \quad = \quad \mathbb{P}\big[S_k \leq t \big| J_k J_{k-1} = \pm 1\big].$$

Remarks

- The sequence $(S_k)_k$ is (unconditionally) i.i.d with distribution:

$$F \quad = \quad \frac{1+\alpha}{2} F_+ + \frac{1-\alpha}{2} F_-.$$

- $h_+ = \frac{1+\alpha}{2} \frac{f_+}{1-F}$ is the intensity function of price jump in the same direction, $h_- = \frac{1-\alpha}{2} \frac{f_-}{1-F}$ is the intensity function of price jump in the opposite direction

Non parametric estimation of jump intensity

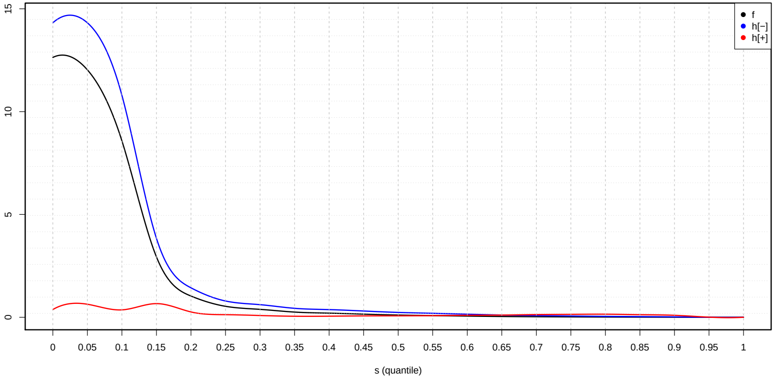


Figure: Estimation of $h_{\pm}$ as function of the renewal quantile

Simulated price

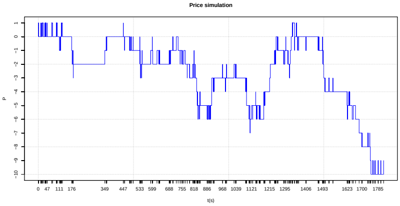


Figure: 30 minutes simulation

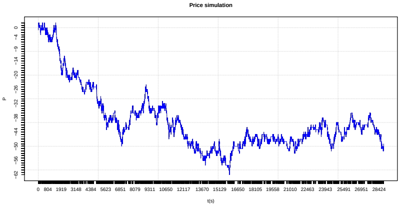


Figure: 1 day simulation

Diffusive behavior at macroscopic scale

Scaling:

$$P_t^{(T)} = \frac{P_{tT}}{\sqrt{T}}, \quad t \in [0,1].$$

Theorem

$$\lim_{T \to \infty} P^{(T)} \stackrel{(d)}{=} \sigma_\infty W,$$

where W is a Brownian motion, and σ_∞^2 is the macroscopic variance:

$$\sigma_\infty^2 = \lambda\Big(\frac{1+\alpha}{1-\alpha}\Big),$$

with $\lambda^{-1} = \int_0^\infty t dF(t)$.

Mean signature plot (realized volatility)

We consider the case of **delayed renewal** process:

- $S_n \rightsquigarrow F$, $n \geq 1$, with finite mean $1/\lambda$, and $S_1 \rightsquigarrow$ density $\lambda(1-F)$

$\rightarrow$ Price process P has stationary increments

Proposition

$$\bar{V}(\tau) \;:=\; \frac{1}{\tau}\mathbb{E}\big[(P_\tau - P_0)^2\big] \;=\; \sigma_\infty^2 + \Big(\frac{-2\alpha}{1-\alpha}\Big)\frac{1-G_\alpha(\tau)}{(1-\alpha)\tau},$$

where $G_\alpha(t) = \mathbb{E}[\alpha^{N_t}]$ is explicitly given via its Laplace-Stieltjes transform $\widehat{G}_\alpha$ in terms of $\widehat{F}(s) := \int_{0^-}^{\infty} e^{-st} dF(t)$.

$$\bar{V}(\infty) \;=\; \sigma_\infty^2, \quad \text{and} \quad \bar{V}(0^+) \;=\; \lambda.$$

Remark: Similar expression as in Robert and Rosenbaum (09) or Bacry et al. (11).

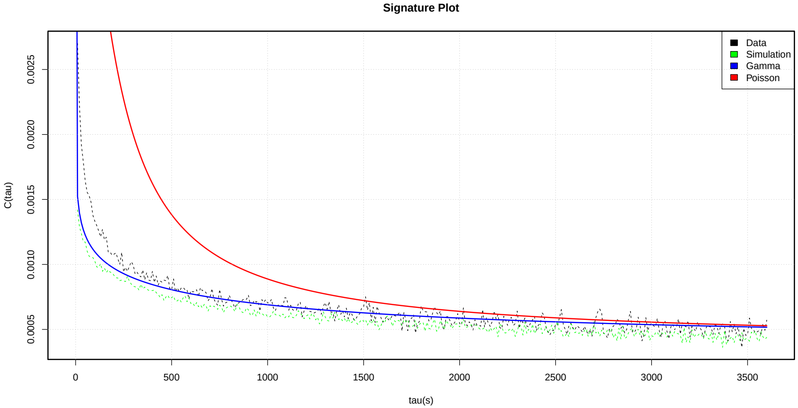


Figure: Mean signature plot for $\alpha < 0$

Markov embedding of price process

- Define the the last price jump direction:

$$I_t = J_{N_t}, \quad t \geq 0, \quad \text{valued in } \{+1, -1\}$$

and the elapsed time since the last jump:

$$S_t = t - \sup_{T_k \leq t} T_k, \quad t \geq 0.$$

▶ Then the price process (P_t) valued in $2\delta\mathbb{Z}$ is embedded in a Markov process with three **observable** state variables (P_t, I_t, S_t) with generator:

$$\mathcal{L}\varphi(p,i,s) = \frac{\partial \varphi}{\partial s} + h_+(s)\big[\varphi(p+2\delta i, i, 0) - \varphi(p,i,s)\big] + h_-(s)\big[\varphi(p-2\delta i, -i, 0) - \varphi(p,i,s)\big],$$

Trading issue

Problem of an agent (market maker) who submits limit orders on both sides of the LOB: limit buy order at the best bid price and limit sell order at the best ask price, with the aim to gain the spread.

- We need to model the market order flow, i.e. the counterpart trade of the limit order

Market trades

- A market order flow is modelled by a marked point process $(\theta_k, Z_k)_k$:
  - θ_k: arrival time of the market order $\leftrightarrow$ M_t counting process
  - Z_k valued in $\{-1, +1\}$: side of the trade.
    - $Z_k = -1$: trade at the best BID price (market sell order)
    - $Z_k = +1$: trade at the best ASK price (market buy order)

index n	θ_k	best ask	best bid	traded price	Z_k
1	9:00:01.123	98.47	98.46	98.47	$+1$
2	9:00:02.517	98.47	98.46	98.46	-1
3	9:00:02.985	98.48	98.47	98.47	-1

▶ Dependence modeling between market order flow and price in LOB

Trade timestamp modeling

- The counting process (M_t) of the market order timestamps $(\theta_k)_k$ is a Cox process with conditional intensity $\lambda_{M}(S_t)$.

Examples of parametric forms reproducing intensity decay when s is large:

$$\begin{aligned} \lambda_M^{exp}(s) &= \lambda_0 + \lambda_1 s^r e^{-ks} \\ \lambda_M^{power}(s) &= \lambda_0 + \frac{\lambda_1 s^r}{1+s^k}. \end{aligned}$$

with positive parameters λ_0, λ_1, r, k, estimated by MLE.

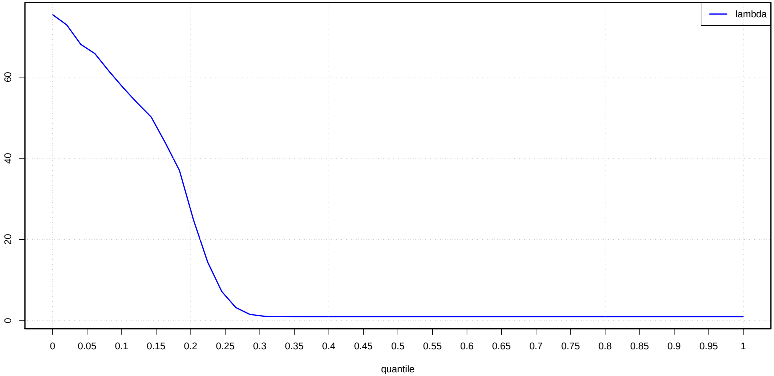
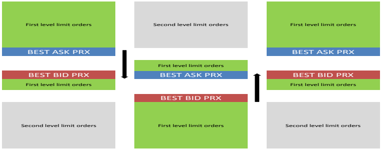


Figure: Estimation of λ_M

Strong and weak side of LOB



• We call strong side (+) of the LOB, the side in the same direction than the last jump, e.g. best ask when price jumped upwards.

• We call weak side (−) of the LOB, the side in the opposite direction than the last jump, e.g. best bid when price jumped upwards.

▶ We observe that trades (market order) arrive mostly on the weak side of the LOB.

Trade side modeling

- The trade sides are given by:

$$Z_k = \Gamma_k \, I_{\theta_k^-},$$

$(\Gamma_k)_k$ i.i.d. valued in $\{+1, -1\}$ with law:

$$\mathbb{P}[\Gamma_k = \pm 1] = \frac{1 \pm \rho}{2}$$

for $\rho \in [-1, 1]$.

Interpretation of ρ

$$\rho = \mathrm{corr}(Z_k, I_{\theta_k^-})$$

- $\rho = 0$: market order flow arrive independently at best bid and best ask (usual assumption in the existing literature)
- $\rho > 0$: market orders arrive more often in the strong side of the LOB
- $\rho < 0$: market orders arrive more often in the weak side of the LOB

• Estimation of ρ: $\hat{\rho}_n = \frac{1}{n}\sum_{k=1}^{n} Z_k I_{\theta_k^-}$ leads to $\rho \simeq -50\%$: about 3 over 4 trades arrive on the weak side.

▶ ρ related to adverse selection

Market making strategy

- Strategy control: predictable process $(\ell_t^+, \ell_t^-)_t$ valued in $\{0, 1\}$
  - $\ell_t^+ = 1$: limit order of fixed size L on the strong side: $+I_{t^-}$
  - $\ell_t^- = 1$: limit order of fixed size L on the weak side: $-I_{t^-}$
- Fees: any transaction is subject to a fixed cost $\varepsilon \geq 0$

▶ **Portfolio process**:

- Cash $(X_t)_t$ valued in $\mathbb{R}$,
- inventory $(Y_t)_t$ valued in a set $\mathbb{Y}$ of $\mathbb{Z}$

Agent execution

- Execution of limit order occurs when:
  - A **market trade** arrives at θ_k on the strong (resp. weak) side if $Z_k I_{\theta_k^-} = +1$ (resp. -1), and with an executed quantity given by a distribution (price time priority/prorata) ϑ_L^+ (resp. ϑ_L^-) on $\{0, \ldots, L\}$
  - The **price jumps** at T_k and crosses the limit order price

Remark
$\vartheta_L^\pm$ cannot be estimated on historical data. It has to be evaluated by a backtest with a zero intelligence strategy.

▶ **Risks:**

- Inventory $\leftrightarrow$ price jump
- Adverse selection in market order trade

Market making optimization

- Value function of the market making control problem:

$$v(t,s,p,i,x,y) = \sup_{(\ell^+,\ell^-)} \mathbb{E}\Big[PNL_T - CLOSE(Y_T) - \eta \cdot RISK_{t,T}\Big]$$

where $\eta \geq 0$ is the agent risk aversion and:

$$\begin{aligned} PNL_t &= X_t + Y_t \cdot P_t, \quad \text{(ptf valued at the mid price)} \\ CLOSE(y) &= -(\delta + \varepsilon) \cdot |y|, \quad \text{(closure market order)} \\ RISK_{t,T} &= \int_t^T Y_u^2 \cdot d[P]_u, \quad \text{(no inventory imbalance)} \end{aligned}$$

Variable reduction to strong inventory and elapsed time

Theorem

The value function is given by:

$$v(t,s,p,i,x,y) \quad = \quad x + yp + \omega_{yi}(t,s)$$

where $\omega_q(t,s) = \omega(t,s,q)$ is the unique viscosity solution to the integro ODE:

$$\left[\partial_t + \partial_s\right]\omega + 2\delta(h^+ - h^-)q - 4\delta^2\eta(h^+ + h^-)q^2 + \max_{\ell\in\{0,1\}, q-\ell L\in\mathbb{Y}} \mathcal{L}_+^\ell \omega + \max_{\ell\in\{0,1\}, q+\ell L\in\mathbb{Y}} \mathcal{L}_-^\ell \omega = 0$$

$$\omega_q(T,s) = -|q|\,(\delta+\epsilon)$$

in $[0,T]\times\mathbb{R}_+\times\mathbb{Y}$.

$$\mathcal{L}^{\ell}_{\pm} = \mathcal{L}^{\ell}_{\pm,M} + \mathcal{L}^{\ell}_{\pm,jump}$$

- favorable execution of random size $\leq L$ by market order

$$\mathcal{L}^{\ell}_{\pm,M}\,\omega := \lambda_{\pm,M}(s)\int \Big[\omega(t,s,q\mp \mathbf{k}\ell) - \omega(t,s,q) + (+\delta-\varepsilon)k\ell\Big]\vartheta^{\pm}_{L}(\mathbf{dk})$$

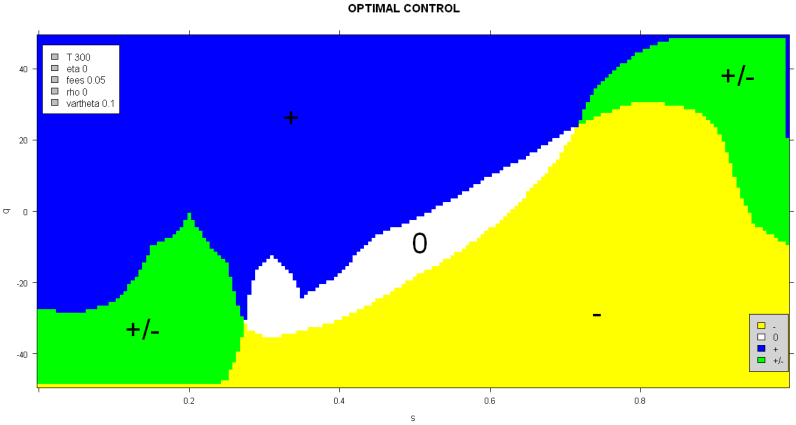
- unfavorable execution of maximal size L due to price jump

$$\mathcal{L}^{\ell}_{\pm,jump}\,\omega := h_{\pm}(s)\Big[\omega(t,0,\pm q-\mathbf{L}\ell) - \omega(t,s,q) + (-\delta-\varepsilon)\mathbf{L}\ell\Big]$$

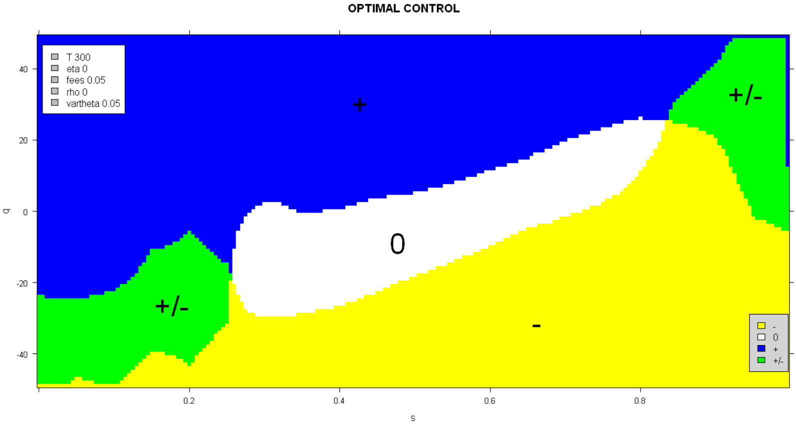
with

$$\lambda_{\pm,M}(s) := \frac{1\pm\rho}{2}\cdot\lambda_{M}(s) \qquad \textit{(trade intensities)}$$

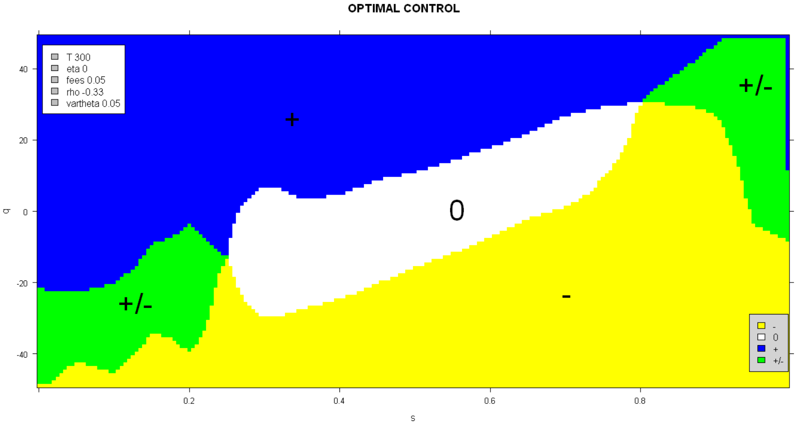
Optimal policy shape: $\rho = 0$, execution probability $= 10\%$



Optimal policy shape: $\rho = 0$, execution probability = 5%



Optimal policy shape: $\rho = -0,33$, execution probability $= 5\%$



Concluding remarks

- Markov renewal approach for market microstructure
  - + Easy to understand and simulate
  - + Non parametric estimation based on i.i.d. sample data
  - + dependency between price return J_k and jump time T_k
  - + Reproduces well microstructure effects, diffuses on macroscopic scale
  - + Markov embedding with observable state variables ($\neq$ Hawkes process approach)
  - + Enables to deal efficiently with trading optimization problem
  - - MRP forgets correlation between inter-arrival jump times $\{S_k = T_k - T_{k-1}\}_k$
- Extension to multivariate price model