Hedging Barrier Options via a General Self-Duality

Thorsten Rheinländer

Vienna University of Technology

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Different concepts of hedging

- **Dynamic hedging:** portfolio gets adjusted continuously

$\oplus$ Typically gives a good approximation to the option price
$\ominus$ Incurs high transaction costs; high model risk for exotic options

- **Static hedging:** hedging instruments get purchased only at inception of the contract

$\oplus$ Minimal transaction costs, often model independent
$\ominus$ Does not reflect the path-dependence of exotic options, therefore gives only a poor hedging performance

- **Semi-static hedging:** trading takes place at inception and at finitely many random times 'when events happen'

$\oplus$ Yields sometimes an exact match to the payoff of certain exotic options by very basic options
$\circledast$ Is to a certain degree model-dependent (to be discussed)

Hierarchy of derivatives

1. **Exotic derivatives:** Barrier options, Asian options, Lookback options, Variance Swaps etc. : Should be converted into simpler ones by dual market principles or *semi-statically* hedged. This sometimes can be achieved by a hedging portfolio of European options.
2. **European options:** Payoff depends only on the price of the underlying asset at maturity. These can be approximated *statically* by a portfolio of simple call and put options as well as futures at various strike levels.
3. **Plain vanilla options:** These can be hedged *dynamically* with the underlying asset.

Dual processes

Definition

Let $S = \exp(X)$ be a martingale with $E[S_T] = 1$. We define the **dual measure** $\widehat{\mathbb{P}}$ by

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} = S_T .$$

The **dual process** $\widehat{S}$ is

$$\widehat{S} = \frac{1}{S} = \exp(-X) .$$

By Bayes' formula, $\widehat{S}$ is a martingale with respect to $\widehat{\mathbb{P}}$.

- Under certain symmetry assumptions, Asian and lookback options with floating and fixed strike are equivalent under duality, see Eberlein, Papapantoleon, Shiryaev (2008).

The Russian option

- Let S be the price process of a risky asset, $r > 0$ the interest rate. The value of the infinite time horizon Russian option (Shepp & Shiryaev (1994)) is

$$V = \sup_{\tau \geq 0} E_{\mathbb{P}} \left[e^{-r\tau} \sup_{0 \leq u \leq \tau} S_u \right]$$

  where the supremum is taken over all stopping times τ.
- Suppose S is a $\mathbb{P}$-martingale and define a consistent family of dual measures $(\mathbb{Q}_T)$ via $d\mathbb{Q}_T / dP = S_T / S_0$ so that there exists a measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_\infty)$ such that the restriction of $\mathbb{Q}$ to $\mathcal{F}_T$ equals $\mathbb{Q}_T$.
- The value of the Russian option can then be written as

$$V = \sup_{\tau \geq 0} E_{\mathbb{P}} \left[S_\tau e^{-r\tau} \sup_{0 \leq u \leq \tau} \frac{S_u}{S_\tau} \right] = \sup_{\tau \geq 0} E_{\mathbb{Q}} \left[e^{-r\tau} \sup_{0 \leq u \leq \tau} \frac{S_u}{S_\tau} \right].$$

- Let M be a continuous $(\mathbb{P}, \mathbb{F})$-martingale vanishing at zero and such that $[M]_\infty = \infty$, and consider its DDS representation $M = B_{[M]}$. The process M is called an **Ocone martingale** if B and $[M]$ are independent.
- Let B, W be two independent Brownian motions. An example of an Ocone martingale is provided by

$$dM = V\,dB \tag{1}$$

where V is $\mathbb{F}^W$-adapted and such that M is a martingale. In this case we say that M is an Ocone SV-model.

- We assume that there exists a weak solution Z to the SDE

$$dZ = dM + \frac{1}{2}\mathrm{sgn}(Z)\,d[M].$$

This is true if M is an Ocone SV-model, but it is doubtful whether it is true for Ocone martingales in general (Vostrikova & Yor (2000)).

- Variation of a theme by Lévy: relates the reflected process $|Z|$ to the drifting process

$$X = M - \frac{1}{2}[M]$$

reflected off its maximum $X^* := \sup X$.

- **Proposition.** Let M be an Ocone SV-model, then

$$|Z| \sim X^* - X.$$

- As a consequence, the value of the Russian option can then be written as

$$\begin{aligned} V &= \sup_{\tau \geq 0} E_{\mathbb{P}} \left[S_\tau e^{-r\tau} \sup_{0 \leq u \leq \tau} \frac{S_u}{S_\tau} \right] \\ &= \sup_{\tau \geq 0} E_{\mathbb{Q}} \left[e^{-r\tau} \sup_{0 \leq u \leq \tau} \frac{S_u}{S_\tau} \right] = \sup_{\tau \geq 0} E_{\mathbb{Q}} \left[e^{-r\tau} e^{|Z_\tau|} \right]. \end{aligned}$$

Overview

- Let the price process be modelled as a continuous stochastic volatility model with correlation.
- Consider a down-and-in call with strike higher than the barrier level, or its up-and-in put analogue.
- We provide a replicating portfolio by trading in stock, realized volatility and cumulative volatility.
- In contrast to market completion by trading in stock and a vanilla option, this does not require to solve a PDE.
- Our method relies on a general self-duality result, whereby duality is to be understood in the sense of dual market; see Eberlein, Papapantoleon and Shiryaev (2008).

Motivation

- Let S be the price process of some risky asset, modelled as a geometric Brownian motion.
- Consider a down-and-in call option with strike price K, maturity T and barrier level $B < K$. We denote $\tau := \inf\{t : S_t \leq B\}$ and assume $S_0 > B$ and that the interest rate is zero.
- If the barrier has been hit before T, the fair price of this option at the barrier is

$$E_\tau^{\mathbb{P}}\left[(S_T - K)^+\right],$$

where $E_\tau^{\mathbb{P}}$ denotes the conditional expectation with respect to the Brownian filtration $(\mathcal{F}_t)$.

- Carr & Chou (1997): This conditional expectation is equal to

$$E_\tau^{\mathbb{P}}\left[\frac{S_T}{B}\left(\frac{B^2}{S_T} - K\right)^+\right].$$

Self-Duality

Definition. *A non-negative adapted process S is* **self-dual** *if for any non-negative Borel function g and any stopping time $\tau \in [0, T]$,*

$$E_\tau^{\mathbb{P}}\left[g\left(\frac{S_T}{S_\tau}\right)\right] = E_\tau^{\mathbb{P}}\left[\left(\frac{S_T}{S_\tau}\right) g\left(\frac{S_\tau}{S_T}\right)\right].$$

- The semi-static replication of the down-and-in call works more generally for continuous self-dual price processes: Carr & Lee (2009), Molchanov & Schmutz (2010). A typical example is a stochastic volatility model where price process and volatility are uncorrelated.

Correlated stochastic volatility models

- Consider the following stochastic volatility model on a time interval $[0, T]$ under a risk-neutral measure $\mathbb{P}$:

$$\begin{aligned} dS_t &= r\,dt + \sigma(V_t)S_t\,dZ_t, \qquad S_0 = s_0 > 0, \\ dV_t &= \mu(V_t)\ dt + \gamma(V_t)\,dW_t, \quad V_0 = v_0 > 0. \end{aligned}$$

- Here Z, W are two Brownian motions with correlation $\rho \in [-1, 1]$. Let $Z = \rho W + \overline{\rho} W^{\perp}$, where W and $W^{\perp}$ are independent standard Brownian motions and $\overline{\rho} = \sqrt{1-\rho^2}$.
- We assume that the functions σ, μ, γ are such that there exists a weak solution (S, V), and that $\sigma(V)$ is non-zero on $[0, T]$. The filtration is set to be $\mathbb{F} = \mathbb{F}^{S,V}$, the filtration generated by S and V.

- Main idea to deal with the *asymmetry risk*: a multiplicative decomposition
$$S = M \times R$$
of the price process S into a self-dual part M and an asymmetric remainder term R.
- We take R_T as Radon-Nikodym derivative to deal with the asymmetry problem via a change of measure:
$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = e^{-rt} R_t, \qquad t \in [0, T].$$
- The **modified price process** D under the measure $\mathbb{Q}$ is defined as
$$D = \frac{S}{R^2} = \frac{M}{R}.$$
- We denote by $\widehat{\mathbb{Q}}$ the dual measure associated with the process D with respect to $\mathbb{Q}$, where
$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{Q}} \Big|_{\mathcal{F}_t} = e^{rt} D_t, \qquad t \in [0, T].$$

General self-duality

- The **general self-duality** holds in our model: for all positive Borel functions g, stopping times $\tau \in [0, T]$,

$$E_\tau^{\mathbb{P}}\left[g\left(\frac{S_T}{S_\tau}\right)\right] = E_\tau^{\widehat{\mathbb{Q}}}\left[g\left(\frac{D_\tau}{D_T}\right)\right],$$

  as well as the **dual general self-duality**:

$$E_\tau^{\mathbb{Q}}\left[g\left(\frac{D_T}{D_\tau}\right)\right] = E_\tau^{\widehat{\mathbb{P}}}\left[g\left(\frac{S_\tau}{S_T}\right)\right].$$

- In the classical self-dual case, self-duality and dual self-duality coincide.

- The fair price of the same down-and-in call as before at the barrier is

$$E_{\tau}^{\mathbb{P}}\left[e^{-r(T-\tau)}\left(S_T-K\right)^+\right].$$

- This expectation is difficult to evaluate in our context. By the general self-duality, this equals $(\tau < T)$

$$E_{\tau}^{\mathbb{Q}}\left[\Gamma_{\tau}^{\mathbb{Q}}\right] = KE_{\tau}^{\mathbb{Q}}\left[e^{-r(T-\tau)}\left(\frac{B}{K}-\frac{D_T}{D_\tau}\right)^+\right].$$

- In contrast to $S_\tau = B$, here D_τ is a random variable. Moreover, D is not a traded instrument, however can be explicitly written as product of S and some functional of the volatility.

Replicating hedging strategy

- Recall that
$$\Gamma_\tau^{\mathbb{Q}} = K\left(\frac{B}{K} - \frac{D_T}{D_\tau}\right)^+ .$$
- We write
$$u\left(x\right) = K\left(\frac{B}{K} - x\right)^+ .$$

By Ito's formula,

$$\begin{aligned} u\left(\frac{D_T}{D_\tau}\right) &= u(1) + \int_\tau^T \frac{\partial u}{\partial x} \cdot \frac{D_t}{S_t}\, dS_t - 2\rho \int_\tau^T \frac{\partial u}{\partial x} D_t \frac{\sigma(V_t)}{\gamma(V_t)} dV_t \\ &- 2r \int_\tau^T \frac{\partial u}{\partial x} D_t\, dt + \int_\tau^T \left(\frac{\partial u}{\partial x} D_t \left(\rho^2 + 2\rho \frac{\mu(V_t)}{\sigma(V_t)\gamma(V_t)}\right)\right. \\ &\left. + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} D_t^2 \left(\frac{1}{S_t^2} + 4\rho^2 - \frac{4\rho^2}{S_t}\right)\right) \sigma^2(V_t)\, dt. \end{aligned}$$

- Finally, substitute

$$D_t = S_t \exp\left(-2rt - 2\rho \int_0^t \frac{\sigma(V_s)}{\gamma(V_s)} dV_s - \int_0^t \frac{\sigma(V_s)\mu(V_s)}{\gamma(V_s)} ds\right) \times \exp\left(\rho^2 \int_0^t \sigma^2(V_s)\, ds\right).$$

- This gives a replicating hedge by dynamically trading in stock, realized variance and bond.
- By using Malliavin calculus, we obtain pricing formulae involving higher greeks.
- Moreover, we give a second order approximation to the price of the barrier option.

Some notation

- $v_t^2 = \frac{1}{T-t}\int_t^T E^{\mathbb{Q}}\left(\sigma_s^2 \middle| \mathcal{F}_t\right)\, ds$. That is, v_t^2 denotes the squared time future average volatility.
- $N_t = \int_0^T E^{\mathbb{Q}}\left(\sigma_s^2 \middle| \mathcal{F}_t\right)\, ds$.
- For all $t < T$, V_t denotes the value at time t of a put option with payoff

$$G(t, D) = K\left(\frac{B}{K} - \frac{D_T}{D_t}\right)^+ .$$

Approximation of barrier option price

- $$\begin{aligned} V_t \approx{}& P_{BS}\left(t, X_t, v_t\right) \\ &+ \frac{\rho}{2} H\left(t, X_t, v_t\right) E^{\mathbb{Q}}\left[\int_t^T e^{-r(s-t)} \sigma_s \, d\left\langle N, W\right\rangle_s \middle| \mathcal{F}_t\right] \\ &+ \frac{K}{8} J\left(t, X_t, v_t\right) E^{\mathbb{Q}}\left[\int_t^T e^{-r(s-t)} \, d\left\langle N, N\right\rangle_s \middle| \mathcal{F}_t\right]. \end{aligned} \tag{2}$$

  Note that, in the above equation, $H\left(t, X_t, v_t\right)$ and $J\left(t, X_t, v_t\right)$ are model-independent and can be written explicitly as:

  $$H\left(t, X_t, v_t\right) = \frac{e^{X_t}}{v_t^2\left(T-t\right)\sqrt{2\pi}} \exp\left(-\frac{d_+^2}{2}\right)\left(-d_-\right)$$

  and

  $$J\left(t, X_t, v_t\right) = \frac{e^{X_t}}{\left(v_t\sqrt{T-t}\right)^3\sqrt{2\pi}} \exp\left(-\frac{d_+^2}{2}\right)\left(d_+ d_- - 1\right).$$

- **Conclusion**: for symmetric continuous SV models, the classical method breaks down in the case there is a significant skewness. One has to hedge also with volatility related instruments.

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