Non-classical BSDEs arising in the utility maximization problem with random horizon

Anthony Réveillac

Work in progress with Monique Jeanblanc and Hai Nam Nguyen

Université Paris Dauphine-CEREMADE

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Motivation

Reference Financial market :

- $W := (W_t)_{t\in[0,T]}$ a Brownian motion defined on $(\Omega, \mathcal{G}, \mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$
- Riskless asset $S^0 := (S^0_t)_{t\in[0,T]}$, zero interest rate
- Risky assets $S := (S_t)_{t\in[0,T]}$,

$$dS_t = S_t\Big(\theta_t dt + dW_t\Big).$$

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Random time : $\tau : \Omega \to I \subset [0, \infty)$ which is $\mathcal{G}$-measurable (not a $\mathbb{F}$-stopping time)

- Enlarged filtration : $\mathbb{G} := (\mathcal{G}_t)_{t\in[0,T]}$, $\mathcal{G}_t := \cap_{\varepsilon>0}\{\mathcal{F}_{t+\varepsilon} \vee \sigma(\mathbf{1}_{\tau\le u},\ u \le t+\varepsilon)\}$.
- We assume the **Immersion hypothesis**$(\mathbb{P})$: Any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale.

Motivation

We study :

- $\sup_{\pi \in \mathcal{A}} \mathbb{E}[U(X^{\pi}_{\tau \wedge T})]$ with

$$X_t^{\pi} = x + \int_0^t \pi_s \frac{dS_t}{S_t} = x + \int_0^t \pi_s (dW_s + \theta_s ds), \; \pi \in \mathcal{A}$$

- $\mathcal{A} := \left\{ \pi \in \mathcal{P}(\mathbb{G}), \; \mathbb{E}\left[\int_0^T |\pi_s|^2 ds \right] < \infty \right\}.$

- Some references : Karatzas, Wang ; Blanchet-Scalliet, El Karoui, Jeanblanc, Martellini ; Bouchard, Pham ; El Karoui, Jeanblanc, Jiao ; Jiao, Pham ; Kharroubi, Lim, Ngoupeyou ;...

Motivation

Additional assumptions on τ : **Density hypothesis** : There exists a map γ which is $\mathcal{F}_t \otimes \mathcal{B}((0,\infty))$-mesurable s.t.

$$\mathbb{P}[\tau > \theta | \mathcal{F}_t] = \int_\theta^\infty \gamma(t,u)du$$

- Proposition 4.4 of [El Karoui, Jeanblanc, Jiao] : The process

$$M_t := \mathbf{1}_{\tau \leq t} - \int_0^{t \wedge \tau} \lambda_s ds, \; t \in [0,T]$$

is a $\mathbb{G}$-martingale and $\mathbb{P}[\tau > t | \mathcal{F}_t] = e^{-\Lambda_t}$.

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2 cases : Set $\Lambda_t := \int_0^t \lambda_s ds$

- $\mathrm{Supp}(\tau) = [0,\infty) \Longrightarrow \Lambda_t < \infty, \ \forall t \leq T, \ \mathbb{P}-a.s.$
- $\mathrm{Supp}(\tau) = [0,T] \Longrightarrow \Lambda_t < \infty, \ \forall t < T, \ \mathbb{P}-a.s.; \ \Lambda_T = +\infty, \ \mathbb{P}-a.s.$.

Our problem : $\sup_{\pi\in\mathcal{A}} \mathbb{E}[U(X^{\pi}_{\tau\wedge T})]$, $U(x) := -\exp(-\alpha x)$, $\alpha > 0$

- Use a technique by Hu, Imkeller, Müller (El Karoui, Rouge ; Kharroubi, Lim, Ngoupeyou).
- Value function :

$$V(t,x) := \text{ess sup}_{\pi} \mathbb{E}\left[U\left(x + \int_{t\wedge\tau}^{T\wedge\tau} \pi_s(dW_s + \theta_s ds)\right) |\mathcal{G}_t\right]$$

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- $V(t,x) = U(x) \underbrace{\text{ess inf}_{\pi} \mathbb{E}\left[\exp\left(-\alpha \int_{t\wedge\tau}^{T\wedge\tau} \pi_s (dW_s + \theta_s ds)\right) |\mathcal{G}_t \right]}_{=:V_t}$,

and

$\hookrightarrow$ $V_{\tau\wedge T} = 1$

$\hookrightarrow$ $V(t, X^{\pi}_t) = V_t U(X^{\pi}_t)$ supermartingale, $\forall \pi$

$\hookrightarrow$ $\exists \pi^*$ s.t. $V(t, X^{\pi^*}_t) = V_t U(X^{\pi^*}_t)$ martingale

Guess : $V_t := \exp(\alpha Y_t)$

$$Y_t = 0 - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} f(s, Y_s, Z_s, U_s) ds - \int_{t\wedge\tau}^{T\wedge\tau} U_s dH_s, \ t \in [0, T]. \tag{1}$$

$\hookrightarrow$ BSDEs with random horizon : Darling, Pardoux.

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Use the conditions :

$\hookrightarrow$ $V_{\tau\wedge T} = 1$

$\hookrightarrow$ $V(t, X_t^\pi) = V_t U(X_t^\pi)$ supermartingale, $\forall \pi$, to get

$\hookrightarrow$ $\exists \pi^*$ s.t. $V(t, X_t^{\pi^*}) = V_t U(X_t^{\pi^*})$ martingale

$$Y_t = 0 - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} \left[\frac{|\theta_s|^2}{2\alpha} + Z_s\theta_s + \lambda_s \frac{1 - e^{\alpha U_s}}{\alpha} \right] ds - \int_{t\wedge\tau}^{T\wedge\tau} U_s dH_s$$

$$Y_t = 0 - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} f(s, Y_s, Z_s, U_s) ds - \int_{t\wedge\tau}^{T\wedge\tau} U_s dH_s, \;\; t \in [0, T].$$

Definition (Solution to (1))

A triplet of processes (Y, Z, U) in $\mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{L}^2$ is a solution to Equation (1) if Relation (1) is satisfied for every t in $[0, \tau \wedge T]$ $\mathbb{P}$-a.s., $Y_t = Y_{T\wedge\tau}$ for $t \geq T \wedge \tau$, $Z_t = U_t = 0$ for $t > T \wedge \tau$ on the set $\{\tau < T\}$, with

$$\mathbb{S}^2 := \left\{ Y \in \mathcal{P}(\mathbb{G}), \; \mathbb{E}[\sup_{t\in[0,T]} |Y_t|^2] < \infty \right\}$$

$$\mathbb{H}^2 := \left\{ Z \in \mathcal{P}(\mathbb{G}), \; \mathbb{E}\left[\int_0^T |Z_t|^2 dt \right] < \infty \right\}$$

$$\mathbb{L}^2 := \left\{ U \in \mathcal{P}(\mathbb{G}), \; \mathbb{E}\left[\int_0^T |U_t|^2 \lambda_t dt \right] < \infty \right\}.$$

$$Y_t = 0 - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} f(s, Y_s, Z_s, U_s)ds - \int_{t\wedge\tau}^{T\wedge\tau} U_s dH_s$$

Theorem (Kharroubi, Lim, Ngoupeyou)

Assume that λ is bounded. If the Brownian BSDE

$$Y_t^b = 0 - \int_t^T f^b(s, Y_s^b, Z_s^b, -Y_s^b)ds - \int_t^T Z_s^b dW_s, \quad t \in [0, T]$$

admits a solution in $\mathbb{S}^2(\mathbb{F}) \times \mathbb{H}^2(\mathbb{F})$ with

$$f^b(t, y, z, u)\mathbf{1}_{t\leq\tau} = f(t, y, z, u)\mathbf{1}_{t\leq\tau}$$

then the process (Y, Z, U) defined below is a solution to (1) :

$$Y_t := Y_t^b \mathbf{1}_{t<\tau}; \quad Z_t := Z_t^b \mathbf{1}_{t\leq\tau}; \quad U_t = -Y_t^b \mathbf{1}_{t\leq\tau}.$$

λ bounded $\Rightarrow \Lambda_T < \infty$, $\mathbb{P}$-a.s. $\Rightarrow$ Supp$(\tau) = [0, \infty)$

$\hookrightarrow$What happens when Supp$(\tau) = [0, T]$ (so when $\Lambda_T = \infty$, $\mathbb{P}$-a.s.) ?

Proposition

Assume that Supp$(\tau) = [0, T]$. If the Brownian BSDE

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then the process (Y, Z, U) defined below is a solution to (1) *:*

$$Y_t := Y_t^b \mathbf{1}_{t < \tau}; \quad Z_t := Z_t^b \mathbf{1}_{t \leq \tau}; \quad U_t = -Y_t^b \mathbf{1}_{t \leq \tau}.$$

$$Y_t = 0 - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} f(s, Y_s, Z_s, U_s)ds - \int_{t\wedge\tau}^{T\wedge\tau} U_s dH_s, \ t \in [0, T]$$

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Proposition

Assume that $Supp(\tau) = [0, T]$. *Let* $A \in L^2$. *If the Brownian BSDE*

$$Y_t^{b,A} = A - \int_t^T f^b(s, Y_s^{b,A}, Z_s^{b,A}, -Y_s^{b,A})ds - \int_t^T Z_s^{b,A} dW_s, \quad t \in [0, T]$$

admits a solution in $\mathbb{S}^2(\mathbb{F}) \times \mathbb{H}^2(\mathbb{F})$ *with* $f^b(t, y, z, u)\mathbf{1}_{t\leq\tau} = f(t, y, z, u)\mathbf{1}_{t\leq\tau}$ *then the process* (Y, Z, U) *defined below is a solution to* (1) *:*

$$Y_t := Y_t^{b,A}\mathbf{1}_{t<\tau}; \quad Z_t := Z_t^{b,A}\mathbf{1}_{t\leq\tau}; \quad U_t = -Y_t^{b,A}\mathbf{1}_{t\leq\tau}.$$

Solutions for $A^1 \neq A^2 \Rightarrow$ two solutions for (1) $\Rightarrow$ 2 value functions $\Rightarrow$ problem illposed.

$$Y_t^{b,A} = A - \int_t^T \left[\frac{|\theta_s|^2}{2\alpha} + Z_s^{b,A}\theta_s + \lambda_s \frac{1 - e^{-\alpha Y_s^{b,A}}}{\alpha} \right] ds - \int_t^T Z_s^{b,A} dW_s$$

$$Y_t^{b,A} = A - \int_t^T \left[\frac{|\theta_s|^2}{2\alpha} + Z_s^{b,A}\theta_s + \lambda_s \frac{1 - e^{-\alpha Y_s^{b,A}}}{\alpha} \right] ds - \int_t^T Z_s^{b,A} dW_s$$

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Toy example : λ, θ, A deterministic ; assume $A > 0$.

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$$\int_0^T \lambda_s \left| \frac{1 - e^{-\alpha Y_s^{b,A}}}{\alpha} \right| ds \geq \frac{1 - e^{-\alpha A/2}}{\alpha} \int_{t_0}^T \lambda_s ds = +\infty, \ \mathbb{P} - a.s.!$$

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So it may happen there is no solution for $A \neq 0$

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First remarked by El Karoui : Section "Pathology" in the BSDE *red book*

BSDEs with exploding coefficient : the linear case

$$Y_t = A - \int_t^T (\varphi_s - \lambda_s Y_s) ds - \int_t^T Z_s dW_s; \quad t \in [0, T],$$

$$Y_t = A - \int_t^T (\varphi_s + \lambda_s Y_s) ds - \int_t^T Z_s dW_s; \quad t \in [0, T],$$

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Definition (Solution)

Let A be an element of L^1 and $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that for any $f(t, \cdot, \cdot)$ is predictable. A pair of predictable processes (Y, Z) is solution to the BSDE

$$Y_t = A - \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad t \in [0, T],$$

if $\mathbb{E}\left[\int_0^T |f(t, Y_t, Z_t)| dt + \left(\int_0^T |Z_s|^2 ds\right)^{1/2}\right] < +\infty$ and the relation above is satisfied for any t in $[0, T]$, $\mathbb{P}$-a.s..

BSDEs with exploding coefficient : the linear case

Proposition

Let A be in L^1 and $\varphi := (\varphi_t)_{t\in[0,T]}$ be an element of $\mathbb{H}^1$. The Brownian BSDE

$$dY_t = (\varphi_t - \lambda_t Y_t)dt + Z_t dW_t; \quad Y_T = A. \tag{2}$$

admits no solution if $A \neq 0$. If $A = 0$, the BSDE (2) may admit infinitely many solutions.

BSDEs with exploding coefficient : the linear case

Proposition

Let A be a given constant and $\varphi := (\varphi_t)_{t\in[0,T]}$ be a deterministic map. We assume that λ is a deterministic function. Then

(i) *If $e^{-\Lambda_t}\int_0^t e^{\Lambda_s}\varphi_s ds$ converges to $C \neq 0$ when t goes to T, then the ODE*

$$dY_t = (\varphi_t - \lambda_t Y_t)dt; \quad Y_T = A.$$

admits no solution if $A \neq C$. If $A = C$, it admits $Y_t = e^{-\Lambda_t}\int_0^t e^{\Lambda_s}\varphi_s ds$ as unique solution.

(ii) *If $e^{-\Lambda_t}\int_0^t e^{\Lambda_s}\varphi_s ds$ converges to 0, the ODE has no solution if $A \neq 0$, and a infinite number of solutions given by $Y_t = e^{-\Lambda_t} - Y_0 + \int_0^t e^{\Lambda_s}\varphi_s ds$ in the case $A = 0$.*

(iii) *If $e^{-\Lambda_t}\int_0^t e^{\Lambda_s}\varphi_s ds$ does not converge, the ODE has no solution.*

BSDEs with exploding coefficient : the linear case

Proposition

Let A be a bounded predictable process and $\varphi := (\varphi_t)_{t\in[0,T]}$ be an element of $\mathbb{H}^1$. The Brownian BSDE

$$dY_t = (\varphi_t + \lambda_t Y_t)dt + Z_t dW_t; \quad Y_T = A.$$

admits no solution unless $A = 0$. In that case the BSDE admits a unique solution.

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admits no solution unless $A = 0$. In that case the BSDE admits a unique solution.

Proposition

Let φ be an element of $\mathbb{H}^1$ and A in L^1. Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing map with $f(0) = 0$. The BSDE

$$Y_t = A - \int_t^T [\varphi_s + \lambda_s f(Y_s)]ds - \int_t^T Z_s dW_s, \quad t \in [0, T]$$

admits no solution if $A \neq 0$.

Coming back to the original equation

Theorem

Let φ be a non-negative bounded predictable process and $\alpha > 0$. The BSDE

$$Y_t = A - \int_t^T [\varphi_s + \frac{\lambda_s}{\alpha}(1 - e^{-\alpha Y_s})]ds - \int_t^T Z_s dW_s, \quad t \in [0, T].$$

admits a solution if and only if $A = 0$. In that case, the solution is unique, Y is bounded and $\int_0^{\cdot} Z_s dW_s$ is a BMO-martingale.

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Corollary

The BSDE

$$Y_t = A - \int_t^T [\frac{|\theta_s|^2}{2\alpha} + Z_s\varphi_s + \frac{\lambda_s}{\alpha}(1 - e^{-\alpha Y_s})]ds - \int_t^T Z_s dW_s, \quad t \in [0, T]. \tag{3}$$

admits a solution if and only if $A = 0$. In that case, the solution is unique, Y is bounded and $\int_0^\cdot Z_s dW_s$ is a BMO-martingale

A final word

Link with a work in progress by Confortola, Fuhrman and Jacod about such BSDEs with multiple jumps but no Brownian part.