**Advanced methods in mathematical finance**

September 3-6, 2013

# Resolving Stochastic PDEs with Backward Doubly Stochastic Differential Equations

Achref BACHOUCH (University of Maine)

Joint work with

M.Anis BEN LASMER (ENIT,Tunisia)

Anis MATOUSSI (University of Maine)

Mohamed MNIF (ENIT,Tunisia)

# Plan

# Plan

1. Numerical scheme for F-BDSDEs
2. Rate of convergence for the BDSDE
3. Numercical scheme for the weak solution of the SPDE
4. Implementation and Numerical tests.

# Plan

1. Numerical scheme for F-BDSDEs
2. Rate of convergence for the BDSDE
3. Numercical scheme for the weak solution of the SPDE
4. Implementation and Numerical tests.

# Plan

1. Numerical scheme for F-BDSDEs
2. Rate of convergence for the BDSDE
3. Numercical scheme for the weak solution of the SPDE
4. Implementation and Numerical tests.

# A quasilinear Stochastic PDE

$$u(t,x) = \phi(x) + \int_t^T [\mathcal{L}u(s,x) + f(s,x,u(s,x),\nabla(u\sigma)(s,x))]ds + \int_t^T g(s,x,u(s,x),\nabla(u\sigma)(s,x)) d\overleftarrow{B}_s, \; 0 \le t \le T.$$

where $u : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^K$ and $\mathcal{L}$ is the second order differential operator given by

$$\mathcal{L} := \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^*)_{i,j} \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x^i} \quad , \quad D_\sigma u := \nabla u \sigma$$

$(B_t)_{0 \le t \le T}$ is a standard Brownian motion.

# SPDEs with BDSDEs

Let $(X_s^{t,x})_{t \leq s \leq T}$ be the solution of the SDE :

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r, \quad t \leq s \leq T$$

Assuming that this SDE has a solution, the couple $(Y_s^{t,x}, Z_s^{t,x})_{t \leq s \leq T}$,where $Y_s^{t,x} = u(s, X_s^{t,x})$ and $Z_s^{t,x} = (\nabla u \sigma)(s, X_s^{t,x})$ verify the BDSDE :

$$\begin{aligned} Y_s =& \phi(X_T^{t,x}) + \int_s^T f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})ds \\ &+ \int_s^T g(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\overleftarrow{dB_r} - \int_s^T Z_r^{t,x} dW_r, \quad t \leq s \leq T. \end{aligned}$$

# Some existing numerical methods to solve SPDEs

Mainly analytic methods, based on time-space discretization :

- Euler finite difference schemes (Gyongy I. and Nualart D., 1995. Gyongy I., 1995).
- Finite elements schemes (Walsh J.B., 2005).
- Spectral Galerkin approximation (Jentzen A and Kloeden P.,2010).

# An other alternative : Probablistic approach, using Monte Carlo method

When $g = 0$ : solve a standard BSDE

- Bally V. (1997).
- Zhang J. (2004).
- Bouchard B.and Touzi N. (2004).
- Gobet E., Lemor J. and Warin X. (2006).

When $g \neq 0$ : We extend the Bouchard-Touzi-Zhang approach to this case.

- Let $(W_t)_{0\leq t\leq T}$ and $(B_t)_{0\leq t\leq T}$ be two independent standard Brownian motions, with values respectively in $\mathbb{R}^d$ and in $\mathbb{R}^l$, defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$.
-$\mathcal{N}$ denotes the class of $\mathcal{P}$-null sets of $\mathcal{F}$. For each $t \in [0, T]$, $T > 0$, we define

$$\mathcal{F} \triangleq \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$$

where for any process $\{\eta_t\}$,
$\mathcal{F}_{s,t}^\eta = \sigma\{\eta_r - \eta_s, s \leq r \leq t\} \vee \mathcal{N}, \mathcal{F}_t^\eta = \mathcal{F}_{0,t}^\eta$.

Let $t \leq s_1 \leq s_2$. For some real number $p \geq 2$ and for any $n \in \mathbb{N}$,

Let $\mathbb{H}^p_n([s_1, s_2])$ denote the set of (classes of $dP \times dt$ a.e. equal) n dimensional progressively measurable processes $\{\psi_u; u \in [s_1, s_2]\}$ satisfying :
(i) $||\psi||^p_{\mathbb{H}^p_n([s_1,s_2])} := E[\int_{s_1}^{s_2} |\psi_u|^p du] < \infty$,
(ii) $\psi_u$ is $\mathcal{F}^t_u$-measurable, for a.e. $u \in [s_1, s_2]$.

We denote similarly by $\mathbb{S}^p_n([s_1, s_2])$ the set of continuous n dimensional processes satisfying :
(i) $||\psi||^p_{\mathbb{S}^p_n([s_1,s_2])} := E[\sup_{s_1 \leq u \leq s_2} |\psi_u|^p] < \infty$,
(ii) $\psi_u$ is $\mathcal{F}^t_u$-measurable, for any $u \in [s_1, s_2]$.

$$\textbf{(H1)} \quad |b(x)-b(x')| + \|\sigma(x)-\sigma(x')\| \le C|x-x'|, \forall x, x' \in \mathbb{R}^d.$$

**(H2)** there exist two constants $K>0$ and $0 \le \alpha < 1$ such that for any $(t_1,x_1,y_1,z_1),(t_2,x_2,y_2,z_2) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^k\times\mathbb{R}^{k\times d}$,

$$\textbf{(i)} |f(t_1,x_1,y_1,z_1)-f(t_2,x_2,y_2,z_2)| \le K\big(\sqrt{|t_1-t_2|}+|x_1-x_2| + |y_1-y_2| + \|z_1-z_2\|\big),$$

$$\textbf{(ii)} \|g(t_1,x_1,y_1,z_1)-g(t_2,x_2,y_2,z_2)\|^2 \le K\big(|t_1-t_2|+|x_1-x_2|^2 + |y_1-y_2|^2\big) + \alpha^2\|z_1-z_2\|^2,$$

$$\textbf{(iii)} |\Phi(x_1)-\Phi(x_2)| \le K|x_1-x_2|,$$

$$\textbf{(vi)} \sup_{0\le t\le T} |f(t,0,0,0)| + ||g(t,0,0,0)|| \le K.$$

$$Y_t = \Phi(X_T^{t,x}) + \int_t^T f(X_s,Y_s,Z_s)ds + \int_t^T g(X_s,Y_s,Z_s)d\overleftarrow{B_s} - \int_t^T Z_s dW_s, 0\le t\le T.$$

A solution of this BDSDE is a pair $(Y,Z)\in\mathbb{S}_k^2([t,T])\times\mathbb{H}_{k\times d}^2([t,T])$ and satisfying this equation.

# Numerical scheme

## The Forward process X : Euler scheme

$\pi : t_0 < t_1 < ... < t_N = T$ is a partition of $[0, T]$ with mesh $|\pi| = h = \max_{1 \le n \le N} |t_n - t_{n-1}|$.
$\Delta W_n = W_{t_{n+1}} - W_{t_n}$, and $\Delta B_n = B_{t_{n+1}} - B_{t_n}$, for $n = 1, ..., N$.
$X^N$ a relative approximation of $X$ at these discretisation times : say it is obtained throught an Euler scheme on the equation satisfied by $X$.
As N goes to infinity, $\sup_{0 \le n \le N} E|X_{t_n} - X^N_{t_n}|^2 \to 0$.
The Euler scheme : Let $x \in \mathbb{R}^d$

$$X^N_0 = x,\ X^N_{t_{n+1}} = X^N_{t_{n+1}} + b(X^N_{t_n})h + \Delta W_n \sigma(X^N_{t_n}).$$

# Numerical scheme

## The Forward-Backward Doubly SDE

The solution $(Y, Z)$ of the F-BDSDE is approximated by $(Y^N, Z^N)$ defined by :

$$Y_{t_N}^N = \Phi(X_T^N),$$

and for $0 \leq n \leq N-1$,

$$Y_{t_n}^N = E_{t_n}[Y_{t_{n+1}}^N + hf(t_n, \theta_n^N)] + g(t_{n+1}, \Theta_{n+1}^N)\Delta B_n],$$

$$hZ_{t_n}^N = E_{t_n}\left[Y_{t_{n+1}}^N \Delta W_n^* + g(t_{n+1}, \Theta_{n+1}^N)\Delta B_n \Delta W_n^*\right],$$

where

$$\theta_n{}^N \ := \ (X_{t_n}{}^N, Y_{t_{n+1}}{}^N, Z_{t_n}{}^N), \Theta_{n+1}^N := (X_{t_{n+1}}^N, Y_{t_{n+1}}^N, E_{t_{n+1}}[Z_{t_n}^N]), \forall n = 0, .., N-1.$$

$*$ denotes the transposition operator and $E_{t_n}$ denotes the conditional expectation over the $\sigma$-algebra $F_{t_n}$.

# Numerical scheme

## continuous-time approximation of the FBDSDE

We define also for all $n = 0, .., N-1$, $(Y^N, Z^N)_{t_n \leq s < t_{n+1}}$ as the solution of the following BDSDE :

$$\begin{cases} dY_s^N = -f(t_n, \theta_n^N)ds - g(t_{n+1}, \Theta_{n+1}^N)\overleftarrow{dB_s} + Z_s^N dW_s, \\ \forall n, Y_{t_n}^N \text{ is given by our numerical scheme.} \end{cases}$$

## Malliavin Calculus for the FBSDE

$F$ is a r.v. of the form $F = \hat{f}(W(h_1), ..., W(h_n), B(k_1), ..., B(k_p))$ with $\hat{f} \in C_b^\infty(\mathbb{R}^{n+p}, \mathbb{R})$, $h_1, ..., h_n \in L^2([0,T], \mathbb{R}^d), k_1, ..., k_p \in L^2([0,T], \mathbb{R}^l)$, where

$$W(h_i) := \int_0^T h_i(s) dW_s, \quad B(k_j) := \int_0^T k_j(s) \overleftarrow{dB_s}.$$

$$D_s F := \sum_{i=1}^n \nabla_i \hat{f}\Big(W(h_1), ..., W(h_n); B(k_1), ..., B(k_p)\Big) h_i(s), 0 \le s \le T,$$

$(D_s F)_s$ is the Malliavin derivative of $F$ w.r.t. $W$. $\mathbb{S}$ is the set of random variables of the above form. For such $F$, we define its norm as :

$$\|F\|_{1,2} := \Big\{ E[F^2] + E\Big[\int_0^T |D_s F|^2 ds\Big]\Big\}^{\frac{1}{2}}.$$

$$\mathbb{D}^{1,2} \triangleq \overline{\mathbb{S}}^{\|\cdot\|_{1,2}}.$$

## Malliavin Calculus for the FBSDE

**(H3(i))** $b \in C_b^1(\mathbb{R}^d, \mathbb{R}^d)$ and $\sigma \in C_b^1(\mathbb{R}^d, \mathbb{R}^{d\times d})$

**(H3(ii))** b$\in C_b^2(\mathbb{R}^d, \mathbb{R}^d)$ and $\sigma \in C_b^2(\mathbb{R}^d, \mathbb{R}^{d\times d})$

**(H3(iii))** $\Phi \in C_b^1(\mathbb{R}^d, \mathbb{R}^k), f \in C_b^1([0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{d\times k}, \mathbb{R}^k)$
and g$\in C_b^1([0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{d\times k}, \mathbb{R}^{k\times l})$

**(H3(iv))** $\Phi \in C_b^2(\mathbb{R}^d, \mathbb{R}^k), f \in C_b^2([0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{d\times k}, \mathbb{R}^k)$
and g$\in C_b^2([0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{d\times k}, \mathbb{R}^{k\times l})$.

## Malliavin calculus on the Forward SDE's

Under **(H3(i))** and **(H3(ii))**, there exists $C > 0$ s.t.

$$E\Big[\sup_{0\le u\le T} ||D_s X_u||^p\Big] \le C(1+|x|^p),$$

$$E\Big[\sup_{s\vee r\le u\le T} ||D_s X_u - D_r X_u||^p\Big] \le C|s-r|(1+|x|^p),$$

$$E\Big[\sup_{0\le u\le T} ||D_r D_s X_u||^p\Big] \le C(1+|x|^{2p}).$$

# Representation results for BDSDEs

## Proposition

*Assume that* **(H1)-(H3)** *hold. Then : For* $t \leq s \leq T$, *we have*

$$D_s Y_s = Z_s,$$

*and*

$$\|Z\|^2_{\mathbb{S}^2_{k\times d}([t,T])} \leq C(1 + |x|^2).$$

*For* $l_1, l_2 \leq d$, $t \leq s \leq T$, *we have*

$$D_s^{l_2} D_t^{l_1} Y_s = D_t^{l_2} Z_s^{l_1},$$

*and*

$$\|D_s^{l_1} Z\|^2_{\mathbb{S}^2_{k\times d}([t,T])} \leq C(1 + |x|^4).$$

# Path regularity

We extend the result of Zhang J.(2004) which concerns the $L^2$-regularity of the martingale integrand $Z$

## Proposition

*Assume that* **(H1)-(H3)** *hold. Then for* $t \leq s \leq u \leq T$*, we have*

$$
\begin{aligned}
E\Big[\sup_{r\in[s,u]}|Y_r - Y_s|^2\Big] &\leq C(1+|x|^2)|u-s|,\\
E\Big[||Z_u - Z_s||^2\Big] &\leq C(1+|x|^2)|u-s|.
\end{aligned}
$$

# Main Result

**Discrete time approximation error and rate of convergence for the BDSDE**

Assume that the hypothesis **(H1)-(H3)** hold, define the error

$$Error_N(Y,Z) := \sup_{0\leq t\leq T} E[|Y_t - Y_t^N|^2] + \sum_{n=0}^{N-1} E[\int_{t_n}^{t_{n+1}} ||Z_{t_n}^N - Z_t||^2 dt],$$

Then there exists a positive constant C (depending on $T$, $K$, $\alpha$, $|b(0)|$, $||\sigma(0)||$, $|f(t,0,0,0)|$ and $||g(t,0,0,0)||$) such that

$$Error_N(Y,Z) \leq Ch(1+|x|^2).$$

# Main tools

Genralized Itô Lemma for BDSDEs.
Define the proxy $\bar{Z}$ on each interval $[t_n, t_{n+1})$ by

$$\bar{Z}_{t_n} = \frac{1}{h} E_{t_n}[\int_{t_n}^{t_{n+1}} Z_s ds].$$

Young inequality.
Gronwall Lemma.
Path regularity.

# Numercical scheme for the weak solution of the SPDE

## Weak solution of SPDEs

Since we work on the hole space $\mathbb{R}^d$, we need to introduce a weight function which is integrable and satisfies $\int_{\mathbb{R}^d}(1+|x|^2)\rho(x)dx < \infty$.

For example, $\rho(x) = e^{-\frac{x^2}{2}}$ or $\rho(x) = e^{-|x|}$.
We add more integrability

$$
\begin{aligned}
(\mathbf{H}_\rho)\ \textbf{(i)} \quad & \int_{\mathbb{R}^d} |\phi(x)|^2 \rho(x)dx < \infty, \\
\textbf{(ii)} \quad & \int_0^T \int_{\mathbb{R}^d} |f(t,x,0,0)|^2 \rho(x)dxdt < \infty, \\
\textbf{(iii)} \quad & \int_0^T \int_{\mathbb{R}^d} |g(t,x,0,0)|^2 \rho(x)dxdt < \infty.
\end{aligned}
$$

# Numercical scheme for the weak solution of the SPDE

## Weak solution of SPDEs

$L^2(\mathbb{R}^d, \rho(x)dx)$ is the wighted Hilbert space,
$(u,v) := \int_{\mathbb{R}^d} u(x)v(x)\rho(x)dx$, $\|u\|_2 := (u,u)^{\frac{1}{2}}$.
$H^1_\sigma(\mathbb{R}^d)$ the associated wighted first order Sobolev space and its norm
$\|u\|_{H^1_\sigma(\mathbb{R}^d)} = (\|u\|_2^2 + \|\nabla u\sigma\|_2^2)^{\frac{1}{2}}$.
$\mathcal{D} := \mathcal{C}_c^\infty([0,T]) \otimes \mathcal{C}_c^2(\mathbb{R}^d)$ is the space of test functions.
$\mathcal{H}_T$ is the space of predictable processes $(u_t)_{t\geq 0}$ valued in $H^1_\sigma(\mathbb{R}^d)$ such that

$$\|u\|_T = \left(E\Big[\sup_{0\leq t\leq T} \|u_t\|_2^2\Big] + E\Big[\int_0^T \|\nabla u_t\sigma\|^2 dt\Big]\right)^{\frac{1}{2}} < \infty.$$

# Numercical scheme for the weak solution of the SPDE

## Definition

We say that $u \in \mathcal{H}_T$ is a weak solution of the SPDE associated with the terminal condition $\phi$ and the coefficients $(f, g)$, if the following relation holds almost surely, for each $\varphi \in \mathcal{D}$

$$\int_t^T (u(s,.), \partial_s \varphi(s,.))ds + \int_t^T \mathcal{E}(u(s,.), \varphi(s,.))ds + (u(t,.), \varphi(t,.)) - (\phi(.), \varphi(T,.))$$

$$= \int_t^T (f(s,.,u(s,.), (\nabla u\sigma)(s,.)), \varphi(s,.))ds + \sum_{i=1}^{l} \int_t^T (g(s,.,u(s,.), (\nabla u\sigma)(s,.)), \varphi(s,.)) \overleftarrow{dB_s^i},$$

where

$$\mathcal{E}(u, \varphi) = (Lu, \varphi) = \int_{\mathbb{R}^d} ((\nabla u\sigma)(\nabla \varphi\sigma) + \varphi\nabla((\frac{1}{2}\sigma^* \nabla\sigma + b)u))(x)dx$$

is the energy of the system associated with the SPDE.

# Numercical scheme for the weak solution of the SPDE

**Theorem**

*Under* $(\mathbf{H1})$,**(H2)**, **(H3)** *and* $(\mathbf{H}_\rho)$, *there exists a unique weak solution* $u \in \mathcal{H}_T$ *of the SPDE associated with the terminal condition* $\Phi$.
*Moreover,* $u(t,x) = Y_t^{t,x}$ *and* $Z_t^{t,x} = \nabla u_t \sigma$, $dt \otimes dx \otimes dP$ *a.e. where* $(Y_s^{t,x}, Z_s^{t,x})_{t \leq s \leq T}$ *is the solution of the BDSDE.*

# Numercical scheme for the weak solution of the SPDE

**Lemma**

*Let $x \in \mathbb{R}^d$ and $t, t_n \in \pi$ such that $t \leq t_n$. Define*

$$u_{t_n}^N(x) := Y_{t_n}^{N,t_n,x} \text{ and } v_{t_n}^N(x) := Z_{t_n}^{N,t_n,x}.$$

*Then $u_{t_n}^N$ (resp. $v_{t_n}^N$) is $\mathcal{F}_{t_n,T}^B$-measurable and we have*

$$u_{t_n}^N(X_{t_n}^{t,x}) = Y_{t_n}^{N,t,x} \quad (\textit{resp. } v_{t_n}^N(X_{t_n}^{t,x}) = Z_{t_n}^{N,t,x}).$$

# Numercical scheme for the weak solution of the SPDE

We define the process $(u_s^N, v_s^N)$ as follows :

$$u_s^N(x) := Y_s^{N,s,x} \text{ and } v_s^N(x) := Z_s^{N,s,x}, \forall s \in [t_n, t_{n+1}).$$

Then

$$u_s^N(X_s^{t,x}) = Y_s^{N,t,x} \text{ and } v_s^N(X_s^{t,x}) = Z_s^{N,t,x}, \forall t \leq s, t, s \in [t_n, t_{n+1}).$$

We define the following error :

$$\begin{aligned} Error_N(u,v) \quad &:= \quad \sup_{0 \leq s \leq T} E_B[\int_{\mathbb{R}^d} |u_s^N(x) - u(s,x)|^2 \rho(x) dx] \\ &+ \quad \sum_{n=0}^{N-1} E_B[\int_{\mathbb{R}^d} \int_{t_n}^{t_{n+1}} \|v_s^N(x) - v(s,x)\|^2 ds \rho(x) dx]. \end{aligned}$$

# Numercical scheme for the weak solution of the SPDE

**Theorem : Rate of convergence for the SPDE**

Assume that $(\mathbf{H1})$,$(\mathbf{H2})$, $(\mathbf{H3})$ and $(\mathbf{H}_\rho)$ hold. Then, the error $Error_N(u,v)$ converges to 0 as $N \to \infty$ and there exists a positive constant C (depending only on $T$, $K$, $\alpha$, $|b(0)|$, $||\sigma(0)||$, $|f(t,0,0,0)|$ and $||g(t,0,0,0)||$) such that

$$Error_N(u,v) \leq Ch.$$

# Notations and algorithm

For each fixed path of $B$, the solution of the BDSDE is approximated by $(Y^N, Z^N)$ : For $0 \leq n \leq N-1$

$\forall j_1 \in \{1, .., k\}$,

$$Y^N_{t_n,j_1} = E_{t_n}\Big[Y^N_{t_{n+1},j_1} + hf_{j_1}(X^N_{t_n}, Y^N_{t_{n+1}}, Z^N_{t_n}) + \sum_{j=1}^{l} g_{j_1,j}(X^N_{t_{n+1}}, Y^N_{t_{n+1}}, E_{t_{n+1}}[Z^N_{t_n}])\Delta B_{n,j}\Big],$$

$\forall j_1 \in \{1, .., k\}$ and $\forall j_2 \in \{1, .., d\}$

$$hZ^N_{t_n,j_1,j_2} = E_{t_n}\Big[Y^N_{t_{n+1},j_1}\Delta W_{n,j_2} + \sum_{j=1}^{l} g_{j_1,j}(X^N_{t_{n+1}}, Y^N_{t_{n+1}}, E_{t_{n+1}}[Z^N_{t_n}])\Delta B_{n,j}\Delta W_{n,j_2}\Big].$$

# Notations and algorithm

## Vector spaces of functions

At every $t_n$, we select $k(d+1)$ deterministic functions bases $(p_{i,n}(.))_{1\leq i\leq k(d+1)}$.
We look for approximations of $Y_{t_n}^N$ and $Z_{t_n}^N$ which will be denoted respectively by $y_n^N$ and $z_n^N$, in the vector space spanned by the basis $(p_{j_1,n}(.))_{1\leq j_1\leq k}$ (respectively $(p_{j_1,j_2,n}(.))_{1\leq j_1\leq k, 1\leq j_2\leq d}$). For example, the hypercube basis.

# Description of the algorithm

→ Initialization : For $n = N$, take $(y_N^{N,m}) = (\Phi(X_{t_N}^{N,m}))$.
→ Iteration : For $n = N-1, ..., 0$ :
• We use $I$ Picard iterations to obtain an approximation of $Z_{t_n}$ :
· For $i = 0$, $\forall j_1 \in \{1,..,k\}$ and $j_2 \in \{1,..,d\}$, $\alpha_{j_1,j_2,n}^{M,0} = 0$.
· For $i = 1,..,I$ : We compute first $E_{t_{n+1}}[Z_{t_n}^N]$ appearing in (1) :

$$\alpha'^{M,i-1}_{j_1,j_2,n+1} = \arg\inf_{\alpha'} \frac{1}{M} \sum_{m=1}^{M} \left| \alpha^{M,i-1}_{j_1,j_2,n} . p^m_{j_1,j_2,n} - \alpha' . p^m_{j_1,j_2,n+1} \right|^2,$$

we set $z'^{N,M,i-1}_{n+1,j_1,j_2}(.) = (\alpha'^{M,i-1}_{j_1,j_2,n+1} . p_{j_1,j_2,n+1}(.))$. After that,

$$\alpha^{M,i}_{j_1,j_2,n} = \arg\inf_{\alpha} \frac{1}{M} \sum_{m=1}^{M} \Bigg| y^{N,M}_{n+1,j_1}(X^{N,m}_{t_{n+1}}) \frac{\Delta W^m_{n,j_2}}{h} + \sum_{j=1}^{l} g_{j_1,j}\Big(X^{N,m}_{t_{n+1}}, y^{N,M}_{n+1}(X^{N,m}_{t_{n+1}}), (\alpha'^{M,i-1}_{j_1,j_2,n+1} . p^m_{j_1,j_2,n+1})\Big) \frac{\Delta B_{n,j} \Delta W^m_{n,j_2}}{h} - \alpha . p^m_{j_1,j_2,n} \Bigg|^2.$$

# Notations and algorithm

## Description of the algorithm

After that we approximate (1) by calculating $\alpha^M_{j_1,n}$, for every $j_1 \in \{1,..,k\}$, as the minimizer of :

$$\frac{1}{M}\sum_{m=1}^{M}\Big|y^{N,M}_{n+1,j_1}(X^{N,m}_{t_{n+1}}) + hf_{j_1}\Big(X^{N,m}_{t_n}, y^{N,M}_{n+1}(X^{N,m}_{t_{n+1}}), z^{N,M,I}_n(X^{N,m}_{t_n})\Big)$$

$$+\sum_{j=1}^{l} g_{j_1,j}\Big(X^{N,m}_{t_{n+1}}, y^{N,M}_{n+1}(X^{N,m}_{t_{n+1}}), z'^{N,M,I}_{n+1}(X^{N,m}_{t_{n+1}})\Big)\Delta B_{n,j} - \alpha p^m_{j_1,k}\Big|^2.$$

Finally, we define $y^{N,M}_n(.)$ as :

$$y^{N,M}_{n,j_1}(.) = (\alpha^M_{j_1,n}.p_{j_1,n}(.)), \forall j_1 \in \{1,..,k\}.$$

# One dimensionnal case (Case when $d = k = l = 1$)

## Case when $f$ and $g$ are linear in $y$ and independant of $z$

$$\begin{cases} dX_t = X_t(\mu dt + \sigma dWt), \\ \Phi(x) = -x + K,\ f(y) = a_0 y,\ g(y) = b_0 y \end{cases}$$

and we set $K = 115$, $r = 0.01$, $R = 0.06$, $X_0 = 100$, $\mu = 0.05$, $\sigma = 0.2$, $T = 0.25$, $d_1 = 60$ , $d_2 = 200$ , $a_0$ and $b_0$ are fixed constants.
Let $Y_{explicit}$ be the solution of our BDSDE in this particular case.
By an integration by parts formula we get

$$Y^{t,x}_{t,explicit} = E[\Phi(X^{t,x}_T)e^{a_0(T-t)+b_0(B_T-B_t)-\frac{1}{2}b_0^2(T-t)}/\mathcal{F}^B_{t,T}]$$

At t=0, we have

$$\begin{aligned} Y^{0,x}_{0,explicit} &= E[\Phi(X^{0,x}_T)e^{(a_0-\frac{1}{2}b_0^2)T+b_0B_T}/\mathcal{F}^B_{0,T}] \\ &= e^{(a_0-\frac{1}{2}b_0^2)T+b_0B_T}E[\Phi(X^{0,x}_T)]. \end{aligned}$$

# One dimensionnal case (Case when $d = k = l = 1$)

## Case when $f$ and $g$ are linear in $y$ and independant of $z$

In the other hand, we compute the solution $Y^{0,x}_{0,explicit}$ in this linear case by using the explicit formula of the expectation of $X^{0,x}_T$,

$$Y^{0,x}_{explicit} = e^{(a_0 - \frac{1}{2} b_0^2)T + b_0 B_T} E[\Phi(X^{0,x}_T)] = e^{(a_0 - \frac{1}{2} b_0^2)T + b_0 B_T} (K - xe^{\mu T})$$

# One dimensionnal case (Case when $d = k = l = 1$)

## Case when $f$ and $g$ are linear in $y$ and independant of $z$

For $a_0 = 0.5$, $b_0 = 0.5$ and $\delta = 1$

N=20, $Y^{0,x}_{explicit} = 13.724$

| $M$ | $\overline{Y}^{0,x,N,M}_0 (\sigma^{N,M})$ | $\frac{\lvert Y^{0,x}_{explicit} - \overline{Y}^{0,x,N,M}_0 \rvert}{Y^{0,x}_{explicit}}$ |
|---|---|---|
| 100 | 13.911(1.178) | 0.013 |
| 1000 | 13.793(0.309) | 0.004 |
| 5000 | 13.848(0.117) | 0.009 |

For $a_0 = 0.5$, $b_0 = 0.5$ and $\delta = 0.5$

N=30, $Y^{0,x}_{explicit} = 14.115$

| $M$ | $\overline{Y}^{0,x,N,M}_0 (\sigma^{N,M})$ | $\frac{\lvert Y^{0,x}_{explicit} - \overline{Y}^{0,x,N,M}_0 \rvert}{Y^{0,x}_{explicit}}$ |
|---|---|---|
| 100 | 14.245(1.045) | 0.009 |
| 1000 | 14.194(0.337) | 0.005 |
| 5000 | 14.235(0.129) | 0.008 |

# One dimensionnal case (Case when $d = k = l = 1$)

**Comparison of numerical approximations of the solutions of the FBDSDE and the FBSDE in the general case**

$$\begin{cases} \Phi(x) = -x + K, \\ f(t, x, y, z) = -\theta z - ry + (y - \frac{z}{\sigma})^{-}(R - r) \\ g_1(t, x, y, z) = 0.1z + 0.5y + log(x) \end{cases}$$

and we set $\theta = (\mu - r)/\sigma$, $K = 115$, $X_0 = 100$, $\mu = 0.05$, $\sigma = 0.2$, $r = 0.01$, $R = 0.06$, $\delta 1 = 1$, $N = 20$, $T = 0.25$ and we fix $d_1 = 60$ and $d_2 = 200$and we set $\theta = (\mu - r)/\sigma$, $K_1 = 95$, $K_2 = 105$, $X_0 = 100$, $\mu = 0,05$, $\sigma = 0,2$, $r = 0,01$, $R = 0,06$, $\delta 1 = 1$,$N = 20$ , $T = 0,25$ and we fix $d_1 = 60$ and $d_2 = 200$.
We finally note that for the contraction constant taken in the following ($\alpha = 0.1$), our algorithm converges after at most three Picard iterations.

## Comparison of numerical approximations of the solutions of the FBDSDE and the FBSDE in the general case : When $t = 0$

| $M$ | $\overline{Y}_{0,BSDE}^{0,x,N,M}(\sigma^{N,M})$ | $\overline{Y}_{0}^{0,x,N,M}(\sigma^{N,M})$ |
|---|---|---|
| 128 | 15.431(1.005) | 13.427(1.175) |
| 512 | 15.029(0.428) | 12.801(0.474) |
| 2048 | 14.763(0.243) | 12.476(0.263) |
| 8192 | 14.718(0.098) | 12.403(0.099) |
| 32768 | 14.715(0.060) | 12.391(0.056) |

# One dimensionnal case (Case when $d = k = l = 1$)

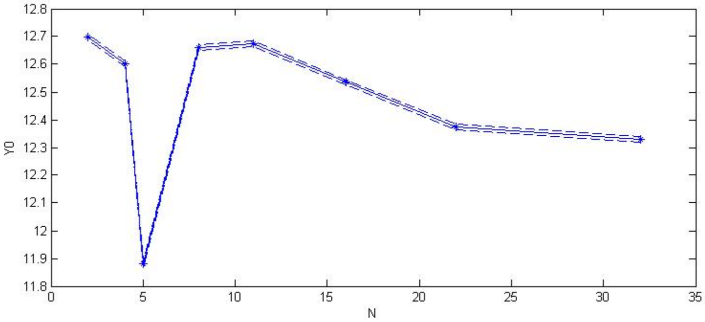


FIGURE : The BDSDE's solution with respect to the number of time discretization steps is with cross markers. Confidence interval are with dotted lines. The figure is obtained for $M = 2000$ and $\delta = 1$.

# One dimensionnal case (Case when $d = k = l = 1$)

Finally, we see on the following figure the impact of the function $g$ on the solution ; we variate $N$, $M$ and $d$ as follows : First we fix $d_1 = 40$ and $d_2 = 180$. Let $j \in \mathbb{N}$, we take $\alpha_M = 3$, $\beta = 1$, $N = 2(\sqrt{2})^{(j-1)}$, $M = 2(\sqrt{2})^{\alpha_M(j-1)}$ and $d = 50/(\sqrt{2})^{(j-1)(\beta+1)/2}$. Then, we draw the map of each solution at $t = 0$ with respect to j.

# One dimensionnal case (Case when $d = k = l = 1$)

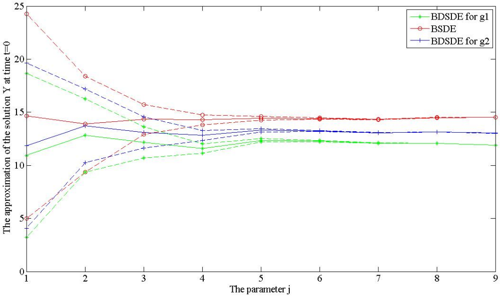


FIGURE : Comparison of the BSDE's solution and the BDSDE's one : The solution of the BSDE is with circle markers, the solution of the BDSDE for $g_1(x, y, z) = 0.1z + 0.5y + log(x)$ is with star markers and the one for $g_2(y, z) = 0.1z + 0.5y$ is with cross markers. Confidence intervals are with dotted lines.

References :


- Pardoux E., Peng S. (1994) : *Backward doubly stochastic differential equations and systems of quasilinear SPDE's*, Probab. Theory Relat. Fields 98, 209-227.
- Bally V., Matoussi A. (1999) : *Weak solutions for SPDEs and Backward Doubly Stochastic Differential Equations*, Theoretical Probability, Vol.14,No.1, 125-164.
- Bouchard B., Touzi N. (2004) : *Discrete time approximation and Monte-Carlo Simulation of Backward Stochastic differential equations*, Stochastic Processes and Their applications, 111, 175-206.
- Gobet, E., Lemor, J.P, Warin X.(2006) : *Rate of convergence of an empirical regression method for solving generalized backward stochastic differential equations*, Bernoulli 12 , no. 5, 889-916

Thank you for your attention !