Semimartingale models with additional information and their applications in Mathematical Finance

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Modelling

- An investor carry out the trading of risky asset $S = \mathcal{E}(X)$, depending on random parameter ξ
- X is a semi-martingale which is also Markov-Feller process given on canonical probability space $(\Omega, \mathcal{F}, P)$
- ξ is random factor which can be a random variable or random process given on a canonical probability space $(\Sigma, \mathcal{H}, \alpha)$
- ξ can represent the additional economic information, for example a price process of a correlated risky asset or default time
- Dependence of the process X on ξ can be given by the family of regular conditional laws $(P^u)_{u\in\Sigma}$: $\forall u \in \Sigma$

$$P^u(X \in \cdot) = P(X \in \cdot | \xi = u)$$

- On product space $(\Omega \times \Sigma, \mathcal{F} \otimes \mathcal{H})$ one extends the probability measure $\mathbb{P}$: $\forall A \in \mathcal{F}$ and $\forall B \in \mathcal{H}$

$$\mathbb{P}(A \times B) = \int_B P^u(A) d\alpha(u).$$

Indifference pricing

- The same investor holds a European type option with pay-off function $G_T = g(\xi)$ which he can not trade because of lack of liquidity or legal restrictions.
- We consider the HARA - utility functions, which are logarithmic, power and exponential utilities :

$$
\begin{aligned}
U(x) &= \log x \\
U(x) &= \frac{x^p}{p}, \; p < 1 \\
U(x) &= 1 - e^{-\gamma x}, \; \gamma > 0.
\end{aligned}
$$

QUESTION What is indifference price for buyer and seller of the option or what is a deterministic amount of money which buyer would like to pay today (and seller would like to receive today) for the right to receive (to transmit) the option at time T and to be indifferent to the situation of the non-having a claim, in the sense that his expected utility will be not changed under the optimal trading strategies in the both situations ?

Utility optimisation

Optimal expected utility with option:

$$V_T(x,g) = \sup_{\phi\in\Pi} E_{\mathbb{P}}[U(x + \int_0^T \phi_s\, dS_s + g(\xi))]$$

- x is initial capital
- $\Pi = \bigcup_{c>0} \{\varphi(\xi) \in \mathcal{P}(\mathbf{F}) \otimes \mathcal{H} \mid \int_0^t \varphi_s(\xi) dS_s \geq -c,\ \forall t \in [0, T]\ (\mathbb{P}\text{-a.s.})\}$

Indifference price for buyer p^b_T is a solution of

$$V_T(x - p^b_T, g) = V_T(x, 0)$$

Indifference price for seller p^s_T is a solution of

$$V_T(x + p^s_T, -g) = V_T(x, 0)$$

Level of information

Level of information about ξ change the class of self-financing admissible strategies which we use for maximisation.

- For non-informed agents, the class self-financing admissible strategies Π related with natural filtration $\mathbf{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ generated by risky asset S.
- for partially informed agents the class of self-financing admissible strategies will be related with progressively enlarged filtration with the process corresponding to ξ.
- For perfectly informed agents the class of self-financing admissible strategies will be related with initially enlarged filtration $\mathbf{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$

$$\mathcal{G}_t = \cap_{s>t}(\mathcal{F}_s \otimes \sigma(\xi))$$

Some remarks

- Often it is sufficient to consider the case of initial enlargement since for $t \in [0, T]$
$$\mathcal{F}_t \subseteq \tilde{\mathcal{F}}_t \subseteq \mathcal{G}_t$$
and
$$\tilde{\mathcal{F}}_T = \mathcal{G}_T$$
- The indifference prices are independent on the level of awareness of investor, since the sets of the equivalent martingale measures coincide at the terminal time T and the "best" martingale measure on the initially enlarged filtration, if it exists, is the same "best" martingale measure on the progressively enlarged filtration.

Main assumptions

- P is the law of X
- P^u is the regular conditional law of X given $\xi = u$
- α^t is the regular conditional distribution of ξ given $\mathcal{F}_t$

ASSUMPTION 1 For all $t \in]0, T]$

$$\alpha^t \ll \alpha$$

ASSUMPTION 2 For all $u \in \Xi$

$$P^u \overset{loc}{\ll} P$$

f-minimal divergence martingale measure

- Function f is a convex conjugate of U obtained by Frenchel-Legendre transform of U:

$$f(y) = \sup_{x>0} \left(U(x) - yx \right).$$

- Two sets of equivalent martingale measures:

$$\mathcal{M}(\mathbf{G}) = \left\{ \mathbb{Q} : \ \mathbb{Q} \overset{loc}{\sim} \mathbb{P} \text{ and } S \text{ is } (\mathbb{Q}, \mathbf{G})\text{-martingale} \right\}.$$

$$\mathcal{M}^u(\mathbf{G}) = \left\{ Q^u : Q^u \overset{loc}{\sim} P^u, \ S \text{ is } (Q^u, \mathbb{F})\text{-martingale and } \mathbb{Q} \in \mathcal{M}(\mathbf{G}) \right\}.$$

DEFINITION We say that $Q^{u,*} \in \mathcal{M}^u(\mathbf{G})$ is f-divergence minimal equivalent martingale measure if under $Q^{u,*}$ the process S given $\xi = u$ is a martingale and

$$E_{P^u}\left[f\left(\frac{dQ^{u,*}_T}{dP^u_T} \right) \right] = \inf_{Q^u} E_{P^u}\left[f\left(\frac{dQ^u_T}{dP^u_T}) \right) \right]$$

Main assumption on existence of f-minimal measure

ASSUMPTION 3 For all $u \in \Xi$ there exists the f-minimal divergence equivalent martingale measure $Q^{u,*} \in \mathcal{M}^u(\mathbf{G})$, such that

$$\frac{dQ_T^{u,*}}{dP_T^u} = z_T^*(\omega, u), \;\; z_T^*(\omega, \cdot) \text{ is } \mathcal{F}_T \otimes \mathcal{H} - \text{measurable}$$

and

$$\int_\Sigma E_{P^u} \left| f(z_T^*(u)) \right| d\alpha(u) < \infty$$

Theorem on existence of f-minimal divergence measure

THEOREM 1 *Let us suppose that Assumptions 1,2 and 3 hold. Then*
(i) There exists f-minimal divergence equivalent martingale measure $\mathbb{Q}^*_T \in \mathcal{M}(\mathbf{G})$ such that

$$\frac{d\mathbb{Q}^*_T}{d\mathbb{P}_T} = Z^*_T(\xi),$$

where

$$Z^*_T(\xi) = \lambda(\xi) z^*_T(\xi)$$

and $\lambda(\xi)$ is $\mathcal{H}$-measurable random variable with

$$\int_\Sigma \lambda(u) d\alpha(u) = 1.$$

(ii) Moreover,

$$-f^{'}\left(\frac{d\mathbb{Q}^*_T}{d\mathbb{P}_T}\right) = x + g(\xi) + \int_0^T \phi^*_s(\omega,\xi) dS_s, \quad \mathbb{Q}^* - a.s., \qquad (1)$$

for some process $\phi^* \in L_{loc}(S, \mathbb{Q}^*)$ such that $\int_0^\cdot \phi^*_s(\omega,\xi) dS_s$ is martingale under $\mathbb{Q}^*$.
(iii) The process ϕ^* is solution to the global utility maximisation problem:

$$V(g,x) = E_{\mathbb{P}}\left[U(x + \int_0^T \phi^*_s \, dS_s + g(\xi))\right].$$

Reduction to conditional utility maximisation problem

From Theorem 1:

$$V(x,g) =$$

$$E_{\mathbb{P}}\left[U(x + \int_0^T \phi_s^*(\xi)\, dS_s + g(\xi))\right] = E_{\mathbb{P}}\left[U\left(-f^{'}(Z_T^*(\xi))\right)\right].$$

Taking the expectation of the RHS given $\xi = u$ we obtain:

$$\begin{aligned} V(x,g) &= \int_\Sigma E_{P^u}\left[U\left(-f^{'}(Z_T^*(u)\right)\right] d\alpha(u) \\ &= \int_\Sigma E_{P^u}\left[U\left(-f^{'}(\lambda(u) z_T^*(u)\right)\right] d\alpha(u) \end{aligned}$$

From Assumption 3 and (ii) of Theorem 5 from Goll and Ruschendorf (2001), it follows that,

$$-f^{'}(\lambda(u) z_T^*(u)) = x + g(u) + \int_0^T \tilde{\phi}^*(u)_s\, dS_s, \quad (2)$$

where $\tilde{\phi}^*(u)$ is an optimal solution for conditional utility optimisation problem.

Dual approach for conditional maximisation problem

Thus,

$$\begin{aligned} V(x,g) &= \int_\Sigma E_{P^u}\left[U(x+\int_0^T \tilde{\phi}^*(u)_s\, dS_s + g(u))\right] \underset{\cdot}{\alpha}(u) \\ &= \int_\Sigma V^u(x,g) d\alpha(u). \end{aligned}$$

THEOREM 3 *Let us suppose that Assumptions 1,2 and 3 hold, $x > \underline{x}$ and $g > 0$, then*

$$V^u(x,g) = E_{P^u}\left[U\left(-f'\left(\lambda_g(u)\frac{dQ_T^{u,*}}{dP_T^u}\right)\right)\right]$$

and $\lambda_g(u)$ is a unique solution of the equation

$$E_{Q^{u,*}}\left[-f'\left(\lambda_g(u)\frac{dQ_T^{u,*}}{dP_T^u}\right)\right] = x + g(u)$$

HARA utilities and information quantities

We introduce three important quantities related with P^u_T and $Q^{u,*}_T$ namely the entropy of P^u with respect to $Q^{u,*}_T$,

$$\mathbf{I}(P^u_T | Q^{u,*}_T) = -E_{P^u}\left[\ln\left(\frac{dQ^{u,*}_T}{dP^u_T}\right)\right],$$

the entropy of $Q^{u,*}_T$ with respect to P^u_T,

$$\mathbf{I}(Q^{u,*}_T | P^u_T) = E_{P^u}\left[\frac{dQ^{u,*}_T}{dP^u_T}\ln\left(\frac{dQ^{u,*}_T}{dP^u_T}\right)\right],$$

and Hellinger type integrals

$$\mathbf{H}^{(q),*}_T(u) = E_{P^u}\left[\left(\frac{dQ^{u,*}_T}{dP^u_T}\right)^q\right],$$

where $q = \frac{p}{p-1}$ and $p < 1$.

Final result for maximisation for HARA utilities

THEOREM 3 *Under the Assumptions 1 and 2 we have the following expressions for* $V_T(x,g)$:

- *If* $U(x) = \ln x$ *then*

$$V_T(x,g) = \int_{\Xi} [\ln(x + g(u)) + \mathbf{I}(P_T^u | Q_T^{u,*})] d\alpha(u)$$

- *If* $U(x) = \frac{x^p}{p}$ *with* $p < 1, p \neq 0$ *then*

$$V_T(x,g) = \frac{1}{p} \int_{\Xi} (x + g(u))^p \left(\mathbf{H}_T^{(q),*}(u) \right)^{1-p} d\alpha(u)$$

- *If* $U(x) = 1 - e^{-\gamma x}$ *with* $\gamma > 0$ *then*

$$V_T(x,g) = 1 - \int_{\Xi} \exp\{-[\gamma(x + g(u)) + \mathbf{I}(Q_T^{u,*} | P_T^u)]\} \, d\alpha(u)$$

Indifference price for power utility

PROPOSITION 5 *In the case of the power utility, the buyer's and seller's indifference prices are defined respectively from the equations:*

$$\int_{\Xi} [(1 - \frac{p_T^b}{x} + \frac{g(u)}{x})^p - 1] \left(\mathbf{H}_T^{(q),*}(u) \right)^{1-p} d\alpha(u) = 0 \tag{3}$$

and

$$\int_{\Xi} [(1 + \frac{p_T^s}{x} - \frac{g(u)}{x})^p - 1] \left(\mathbf{H}_T^{(q),*}(u) \right)^{1-p} d\alpha(u) = 0 \tag{4}$$

Moreover, under $g(\xi) \in]0, x[$ (α-a.s.) and some integrability conditions, the above equations have unique solutions.

Indifference price for exponential utility

PROPOSITION 6 *In the case of the exponential utility the buyer's and seller's indifference prices verify:*

$$p_T^b = \frac{1}{\gamma} \ln \left[\frac{\int_{\Xi} \exp\left\{ - \mathbf{I}(Q_T^{u,*} | P_T^u) \right\} d\alpha(u)}{\int_{\Xi} \exp\left\{ - \gamma g(u) - \mathbf{I}(Q_T^{u,*} | P_T^u) \right\} d\alpha(u)} \right] \quad (5)$$

and

$$p_T^s = -\frac{1}{\gamma} \ln \left[\frac{\int_{\Xi} \exp\left\{ - \mathbf{I}(Q_T^{u,*} | P_T^u) \right\} d\alpha(u)}{\int_{\Xi} \exp\left\{ \gamma g(u) - \mathbf{I}(Q_T^{u,*} | P_T^u) \right\} d\alpha(u)} \right] \quad (6)$$

Risk measures

The application $\rho : \mathcal{F}_T \to \mathbb{R}^+$ is convex risk measure if for all contingent claims $C_T^{(1)}, C_T^{(2)} \in \mathcal{F}_T$ and all $0 < \gamma < 1$ we have:

1. convexity of ρ with respect to the claims:

$$\rho(\gamma\, C_T^{(1)} + (1-\gamma)\, C_T^{(2)}) \leq \gamma\rho(C_T^{(1)}) + (1-\gamma)\rho(C_T^{(2)})$$

2. it is increasing function with respect to the claim:

$$\text{for } C_T^{(1)} \leq C_T^{(2)}, \text{ we have } \rho(C_T^{(1)}) \leq \rho(C_T^{(2)})$$

3. it is invariant with respect to the translation: for $m > 0$

$$\rho(C_T^{(1)} + m) = \rho(C_T^{(1)}) + m$$

Indifference price and risk measure properties

PROPOSITION 7 *For HARA utilities the indifference prices for sellers* $p^s_T(g)$ *and* $(-p^b_T)$ *for buyers are risk measures.*

How it works: BS models

- Two risky assets

$$S_t^{(1)} = \exp\{(\mu_1 - \frac{\sigma_1^2}{2})t + \sigma_1 W_t^{(1)}\}$$

$$S_t^{(2)} = \exp\{(\mu_2 - \frac{\sigma_2^2}{2})t + \sigma_2 W_t^{(2)}\}$$

with $(W^{(1)}, W^{(2)})$ bi-dimensional standard Brownian motions with correlation ρ, $|\rho| < 1$ on $[0, T]$.

- What is ξ?

$$\xi = W_{T'}^{(2)}$$

- What is X?

$$X_t = \mu_1 t + \sigma_1 W_t^{(1)}$$

Conditional law of X : Assumption 2

- The conditional law of X given $\xi = u$ coincide with the law of

$$X_t = \mu_1 t + \sigma_1 \rho V_t + \sigma_1 \sqrt{1-\rho^2}\gamma_t$$

where V is a Brownian bridge starting from 0 at $t = 0$ and ending in u at $t = T'$ which is independent from Brownian motion γ.

- As known,

$$V_t = \int_0^T \frac{u - V_s}{T' - s} ds + \eta_t$$

where η is standard Brownian motion independent from γ.

- Since $\hat{\gamma} = \rho\eta + \sqrt{1-\rho^2}\gamma$ is again standard Brownian motion, we get:

$$X_t = \mu_1 t + \sigma_1 \rho \int_0^t \frac{u - V_s}{T' - s} ds + \sigma_1 \hat{\gamma}_t$$

- Hence, $P_t^u \ll P_t$ for all $u \in \mathbb{R}$ and $t \in [0, T]$.

Conditional law of ξ : Assumption 1

- We recall that $\xi = W^{(2)}_{T'}$ and $\mathcal{F}_t = \sigma(W^{(1)}_s, s \leq t)$.
- By Markov property we get: for $A \in \mathcal{B}(\mathbb{R})$

$$P(\xi \,|\, \mathcal{F}_t)(A) = P(W^{(2)}_{T'} \in A \,|\, \mathcal{F}_t) = P(W^{(2)}_{T'} \in A \,|\, W^{(1)}_t)$$

$$= P(W^{(2)}_{T'} - W^{(2)}_t + W^{(2)}_t \in A \,|\, W^{(1)}_t)$$

- Finally,

$$P(\xi \,|\, \mathcal{F}_t) = \mathcal{N}(\rho x, T' - \rho^2 t)$$

and since $T' - \rho^2 t \neq 0$ for $t \in [0, T]$, it is equivalent to the law of $W^{(2)}_{T'}$ being $\mathcal{N}(0, T')$.

PROPOSITION 8 *For mentioned three information quantities we have the following result:*

$$\mathbf{I}(P^u \mid Q^{*,u}) = \frac{\sigma_1^2}{2}\left[\left(\mu_1 - \frac{\sigma_1 \rho u}{T'}\right)^2 T + \frac{\sigma_1^2 \rho^2}{T'}\left(T' \ln(\frac{T'}{T'-T}) - T\right)\right],$$

$$\mathbf{I}(Q^{*,u} \mid P^u) = \frac{\sigma_1^2}{2}\Big\{\mu_1^2\, T + 2\sigma_1\, \mu_1\, \rho\, u\, \ln(\frac{T'}{T'-T}) + \sigma_1^2 \rho^2\, u^2 \frac{T}{T'(T'-T)}$$

$$+\sigma_1^2 \rho^2 \left[\frac{T}{T'-T} - \ln(\frac{T'}{T'-T})\right]\Big\},$$

$$\mathbf{H}_T^{(q)}(u) = \left(\frac{T'}{T'-T+qT}\right)^{1/2} \exp\left\{-\frac{(1-q)}{2}\left[\frac{u^2}{T'} - \frac{(u+cT)^2}{T'-T+qT}\right]\right\}$$

with $q > -(\frac{T'}{T} - 1)$ *and* $c = \frac{\mu_1}{\sigma_1 \sqrt{1-\rho^2}}$

Example of two independent Levy processes

- Two independent geometric Brownian motions such that

$$S_t^{(1)} = \exp\{(\mu_1 - \frac{\sigma_1^2}{2})t + \sigma_1 W_t^{(1)}\}$$

$$S_t^{(2)} = \exp\{(\mu_2 - \frac{\sigma_2^2}{2})t + \sigma_2 W_t^{(2)}\}$$

- For simplicity of calculations we consider that $\mu_{(\cdot)} = 0$ and $\sigma_{(\cdot)} = 1$.
- The random variable is a default time $\tau = \inf\left\{t \in [0, T] : S_t^2 \leq a\right\}$.
- We consider that investor buys the option with payoff function $g\left(\mathbb{I}_{\{\tau \leq T\}}\right) = b\mathbb{I}_{\{\tau \leq T\}}$.
- Let the initial capital x be equal to 1, then $b < 1$.
- The distribution of τ is

$$F_\tau(t) = \Phi\left(\frac{\ln a + \frac{T}{2}}{\sqrt{T}}\right) + \frac{1}{a}\Phi\left(\frac{\ln a - \frac{T}{2}}{\sqrt{T}}\right).$$

Example of two independent Levy processes

For the defaultable model one gets the following integral equations for the buyer's indifference price:

- In the case of logarithmic utility:

$$\ln\left(1 - p_T^b + k\right) F_\tau(T) + \ln\left(1 - p_T^b\right)(1 - F_\tau(T)) = 0 \qquad (7)$$

- In the case of power utility, $p < 1, p \neq 0$:

$$\left(\left(1 - p_T^b + k\right)^{p-1} - 1)F_\tau(T) - \frac{1}{2} + \left(\left(1 - p_T^b\right)^{p-1} - 1\right)(1 - F_\tau(T)) = 0 (8)$$

- In the case of exponential utility, $\gamma > 0$:

$$p_T^b = -\frac{1}{\gamma} \ln\left(e^{-\gamma b} F_\tau(T) + 1 - F_\tau(T)\right) \qquad (9)$$

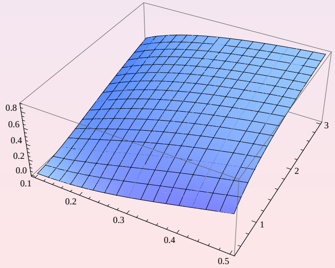
Distribution of τ

We assume $a \in [0.1, 0.5]$ and $T \in [1, 3]$.

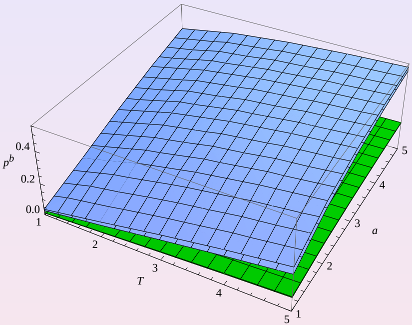
Table: $F_\tau(T)$

Case	$T=1$	$T=1.5$	$T=2$	$T=2.5$	$T=3$
$a=0.1$	0.06107412‘	0.16589305‘	0.27615169‘	0.37604460‘	0.46221476‘
$a=0.2$	0.22088765‘	0.37653772‘	0.49579569‘	0.58641865‘	0.65635072‘
$a=0.3$	0.38803513‘	0.53980954‘	0.64120586‘	0.71270390‘	0.76533803‘
$a=0.4$	0.53446163‘	0.66308077‘	0.74286328‘	0.79690473‘	0.83569574‘
$a=0.5$	0.65623355‘	0.7571794‘	0.81710178‘	0.85673907‘	0.88477023‘

Figure: The distribution of τ



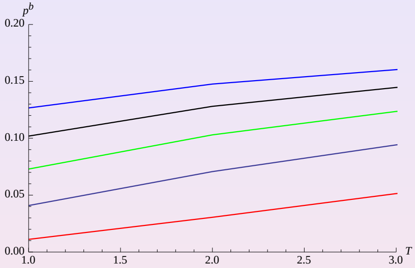
Exponential indifference prices



The exponential utility indifference prices for $\gamma \in (0.1, 2)$.

The corresponding values of the axes T and a are from the grid $[5 \times 5]$ of Table 1.The blue sheets corresponds to the case of $b = 0.2$ and the green sheets to $b = 0.6$. The different layers of the sheets correspond to the different coefficient of risk aversion $\gamma > 0$.

Exponential indifference prices



The exponential utility indifference prices in the case $\gamma = 1$

Red:=$\{a = 0.1\}$

Grey:=$\{a = 0.2\}$

Green:=$\{a = 0.3\}$

Black:=$\{a = 0.4\}$

Blue:=$\{a = 0.5\}$

Numerical result for indifference prices

Table: Indifference prices

Case $a = 0.1,\ b = 0.2$	$p_T^{b,exp},\ \gamma = 1$	$p_T^{b,log}$	$p_T^{b,1/2}$	$p_T^{b,-1/2}$
$T = 1$	0.0111326	0.0111871	0.0107143	0.00984339
$T = 1.5$	0.0305353	0.0306383	0.0294511	0.0272343
$T = 2$	0.0513541	0.0514628	0.0496708	0.0462718
$T = 2.5$	0.0705999	0.0706757	0.0684823	0.0642565
$T = 3$	0.0875046	0.087533	0.0851196	0.0804011

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