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General switching game and related system of Variational inequalities

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Outline of the talk

I- Motivation of the problem

- Preliminary notations
- The switching problem : Presentation and review of existing literature
- The switching game : formulation and objectives

II- Study of the related system of variational inequalities

- Main system : presentation and first (comparison) result
- Presentation of approximating schemes :
- Existence of continuous viscosity solutions (Perron's method)

III- The switching game

- Preliminaries : Min-max and Max-min PDEs and connection with zero sum Dynkin games
- The main result : characterization of the value function

Presentation of the problem

Introduction : Setting and notations

On a standard probability space,

- W : standard d-dim. Brownian Motion,
- X diffusion process s.t. $dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$
  $+$ standard conditions on b, σ,
- T finite horizon $+$ $\mathcal{J} = \{1, \cdots, m\}$ set of possible modes.

1. $\Psi^i(s, X_s)$: instantaneous profit (generated in mode i, i in $\mathcal{J}$)
2. $h^i(X_T)$: fixed payoff (or terminal condition) at time T,
3. $\underline{g}_{i,k}(s, X_s)$: nonnegative penalty costs incurred at time s when system switches from i to k.

Review on standard switching problem

Presentation

- $\mathcal{A}^i$: set of admissible strategies $\alpha := (\tau_p,\ i_p)$ $\tau_0 = 0,\ i_0 = i$ satisfying both

$$\mathbb{P}(\{\forall\ p \in \mathbb{N},\ \tau_p < T\}) = 0$$

and $A_T(\alpha) = \sum_{p\geq 0} g_{i_p,i_{p+1}}(\tau_p, X_{\tau_p})\mathbf{1}_{\tau_p<T}$ square integrable.

- Profit functional (associated with α)

$$J^i(\alpha) = \mathbb{E}\left(h^i(X_T) + \int_0^T \sum_{p\geq 0} \Psi^{i_p}(s, X_s)\mathbf{1}_{s\in[\tau_p,\tau_{p+1}[}ds - A_T(\alpha)\right).$$

Review on standard switching problem

Presentation

- Dynamic version of switching problem (t given in $[0,T]$)
  $\mathcal{A}^{t,i}$: set of admissible strategies s.t. $\tau_0 = t$, $i_0 = i$
  For any α in $\mathcal{A}^{t,i}$, we define

$$J^i(t,\alpha) = \mathbb{E}_{\mathcal{F}_t}\left(h^i(X_T^{t,x}) + \int_t^T \sum_{p\geq 0} \Psi^{i_p}(r, X_r^{t,x})\mathbf{1}_{r\in[\tau_p,\tau_{p+1}[}dr - A_{t,T} \right)$$

with $A_{t,T} = \sum_{p\geq 0} g_{i_p,i_{p+1}}(\tau_p, X_{\tau_p}^{t,x})\mathbf{1}_{t\leq\tau_p<T}$.

- Objectives of switching problem
  - Characterize $V_i = v_i(t,x) = \text{ess sup}_{\alpha\in\mathcal{A}^{t,i}} J^i(t,\alpha)$,
  - Identify and construct α^* achieving the supremum (in $\mathcal{A}^{t,i}$).

The switching problem : the BSDE approach

- The general m modes switching problem :
  Define $(Y^i)_{i\in\{1,..,m\}} = \mathbb{R}^m$-valued process s.t.

$$(\mathcal{S}) \begin{cases} Y^i, K^i,\ Z^i \text{ and } K^i \text{non-decreasing and } K^i_0 = 0; \\ Y^i_s = h_i(X^{t,x}_T) + \int_s^T \Psi_i(r, X^{t,x}_r, Y^1_r, \dots, Y^m_r, Z^i_r)dr \\ \ + K^i_T - K^i_s - \int_s^T Z^i_r dB_r,\ \forall\ s \le T \\ Y^i_s \ge \max_{k\neq i}\{Y^k_s - \underline{g}_{i,k}(s, X^{t,x}_s)\},\ \forall s \le T \\ \int_0^T (Y^i_s - \max_{k\neq i}\{Y^k_s - \underline{g}_{i,k}(s, X^{t,x}_s)\})dK^i_s = 0. \end{cases} \quad (1)$$

$(\mathcal{S})$: system of m reflected BSDEs with interconnected lower obstacle.

Review on standard switching problem

List of hypotheses for the data of the RBSDE system

H1 Ψ_i is uniformly Lipschitz continuous w.r.t. $(\overrightarrow{y}, z^i) := (y^1, ..., y^m, z^i)$, $(s, x) \mapsto \Psi_i(s, x, 0, 0)$ has at most polynomial growth (w.r.t x) (it belongs to the class Π^g)

H2 *Monotonicity* $\forall i \in \mathcal{J}$, $\forall k \in \mathcal{J} \setminus i$, the mapping $y_k \in \mathbb{R} \mapsto \Psi_i(t, x, y_1, ..., y_{k-1}, y_k, y_{k+1}, ..., y_m)$ is non-decreasing whenever $(t, x, y_1, ..., y_{k-1}, y_{k+1}, ..., y_m)$ are fixed.

H3 (i) g_{ij} is jointly continuous in (t, x), non-negative and belongs to Π^g ;

List of hypotheses (continued)

H3 **Non free loop property** (ii) for any $(t, x) \in [0, T] \times \mathbb{R}^k$ and for any sequence $i_1, ..., i_k$ such that $i_1 = i_k$ and $\mathrm{card}\{i_1, ..., i_k\} = k - 1$ we have :

$$g_{i_1 i_2}(t,x) + g_{i_2 i_3}(t,x) + \cdots + g_{i_{k-1} i_k}(t,x) + g_{i_k i_1}(t,x) > 0,$$

$\forall (t, x) \in [0, T] \times \mathbb{R}^k$.

H4 h_i is continuous, belongs to Π^g and satisfies :

$$\forall x \in \mathbb{R}, \quad h_i(x) \geq \max_{j \in \mathcal{J} \setminus i} (h_j(x) - g_{ij}((T, x)).$$

First result for the switching problem

Under assumptions $(\mathbf{Hi})_{i=1,\cdots,4}$, there exists m triples $(Y^i, Z^i, K^i)_i$ satisfying $(\mathcal{S})$.
In addition the following representation holds

$$\forall\ t \in [0, T] \ \ Y^i_t = \text{ess sup}_{\alpha \in \mathcal{A}^{t,i}} J(t, \alpha),$$

the optimal admissible strategy $\alpha^* = (\tau^*_p, i^*_p)$ exists s.t.

$$\tau^*_0 = t,\ \tau^*_p = \inf\{u > \tau^*_{p-1},\ Y^i_u = \max_{k \neq i} \left(Y^k_u - \underline{g}_{i,k}(u, X^{t,x}_u) \right)\}$$

and

$$i^*_0 = i,\ i^*_p = \mathrm{Argmax}\{k,\ Y^{i^*_{p-1}}_{\tau^*_p} = \max \left(Y^k_{\tau^*_p} - \underline{g}_{i,k}(\tau^*_p, X^{t,x}_{\tau^*_p}) \right)\}$$

Solution of the switching problem

Second result for the switching problem

In the Markovian setting (i.e. when randomness of Ψ_i, $(h_i)_{i\in\mathcal{J}}$ and $((\underline{g}_{i,k})_{k\neq i})$ comes from $X = X^{t,x}$)
the family $(v_i : (t,x) \mapsto Y_t^{i,t,x})_{i\in\mathcal{J}}$ is the unique continuous viscosity solution of

$$\begin{cases} \min\Big\{ v_i(t,x) - \max\limits_{j\in\mathcal{J}^{-i}} (-g_{i,j}(t,x) + v_j(t,x)) \,; \\ -\partial_t v_i(t,x) - \mathcal{L}^X v_i(t,x) - \Psi_i(t,x,(v_l(t,x))_l, (\sigma^\top . D_x v_i)(t,x)) \Big\} \\ v_i(T,x) = h_i(x). \end{cases} \quad (2)$$

with
$\mathcal{L}\varphi(t,x) = b(t,x)^T D_x\varphi(t,x) + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(t,x) D_{xx}\varphi(t,x))$,
for φ in $\mathcal{C}^{1,2}([0,T]\times\mathbb{R})$.

The switching problem : Review of existing results

2.1 First studies : Two-modes switching problem (constant penalty costs or non random data). Dixit (1987), Zervos (2006) Ludkowski (phD thesis 2005)

2.2 Generalizations :

- Relationship between the 2-modes switching problem and an explicit doubly reflected BSDE (Hamadène-Jeanblanc - 2002)
- The multi-modal switching problem : Connection with system of obliquely reflected BSDEs
  Hu-Tang (2007), Hamadène-Djehiche-Popier (2008), Ma-Pham-Kharroubi (2008)
  Hamadène Zhang (2010), Elie Kharroubi (2009, 10)
  Chassagneux-Elie-Kharroubi (2011) Hamadene Morlais (2012)
  .
- Numerical aspects : Ludkowski, Elie-Kharroubi (2010) Bernhard (phD 2011)

Presentation of the switching game

Same brownian setting, T fixed time horizon, set of modes $\Gamma = \Gamma^1 \times \Gamma^2$

The gain functional

Assume that
Player 1 has strategy $\alpha = (i_k, \sigma_k)$,
Player 2 has strategy $\beta = (j_k, \tau_k)$
s.t. system is in state (i_k, j_k) during $[\nu_k, \nu_{k+1}[$, $(i_0, j_0) = (i, j)$ then

$$\begin{aligned} J^{i,j}(\alpha, \beta) &= \mathbb{E}\left(h(X_T) + \int_0^T \sum_{k\geq 0} \Psi^{i_k, j_k}(s, X_s) \mathbf{1}_{s\in[\nu_k, \nu_{k+1}[} ds \right) \\ &- \sum_{k\geq 1}\left(\underline{g}_{i_{k-1}, i_k} \mathbf{1}_{\{\nu_k = \sigma_k, \nu_k < T\}} - \bar{g}_{j_{k-1}, j_k} \mathbf{1}_{\{\nu_k = \tau_k,\ \nu_k < T\}} \right) \end{aligned}$$

Objectives of the switching game

(i) Justifying existence to the value function $V = V^{i,j}$

$$V^{i,j} = \sup_{\alpha \in \mathcal{A}^i} \inf_{\beta \in \mathcal{B}^j} J^{i,j}(\alpha, \beta) = \inf_{\beta} \sup_{\alpha} J^{i,j}(\alpha, \beta)$$

(ii) Characterizing an optimal mixed strategy (when it exists !) as a saddle point

$$\forall\ (\alpha, \beta), \quad J^{i,j}(\alpha, \beta^*) \leq J^{i,j}(\alpha^*, \beta^*) \leq J^{i,j}(\alpha^*, \beta)$$

Second part : Related system of variational inequalities

Study of related system of variational inequalities

The main system

For any $(i,j) \in \Gamma^1 \times \Gamma^2$

$$\begin{cases} \min\Big\{ \big(v^{i,j} - L^{i,j}[\vec{v}]\big)(t,x)\,; \\ \max\Big\{\big(v^{i,j} - U^{i,j}[\vec{v}]\big)(t,x); \\ \qquad -\partial_t v^{i,j}(t,x) - \mathcal{L}v^{i,j}(t,x) - \Psi^{i,j}(t,x,(v^{k,l}(t,x)))\Big\}\Big\} = 0 \\ v^{i,j}(T,x) = h_{i,j}(x) \end{cases} \tag{3}$$

where for any (t,x),

$$\mathcal{L}\varphi(t,x) = b(t,x)D_x\varphi(t,x) + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^T(t,x)D^2_{xx}\varphi(t,x)],$$

$$L^{i,j}[\vec{v}](t,x) := \max_{k\in(\Gamma^1)^{-i}}(v^{k,j}(t,x) - \underline{g}_{i,k}(t,x))$$

$$U^{i,j}[\vec{v}](t,x) = \min_{l\in(\Gamma^2)^{-j}}(v^{i,l}(t,x) + \bar{g}_{j,l}(t,x)).$$

Study of related system of variational inequalities

The main system : hypotheses

1. for any (i,j), $\Psi^{i,j}$ Lipschitz w.r.t. $\vec{y}$ (uniformly in (t,x,z),
2. Monotonicity : for $(k,l) \neq (i,j)$, $y^{k,l} \mapsto \Psi^{i,j}(t,x,\vec{y})$ non decreasing,
3. $\Psi^{i,j}$ may depend on $\vec{z}$ only through $z^{i,j}$.
4. Constraints on terminal conditions

$$\max_{k\in(\Gamma^1)^{-i}}(h^{k,j}(x) - \underline{g}_{i,k}(T,x)) \leq h^{i,j}(x) \leq \min_{l\in(\Gamma^2)^{-j}}(h^{i,l}(x) + \bar{g}_{j,l}(T,x))$$

5. + Technical conditions on penalty costs $(\underline{g}_{i,k})_{k\neq i}$ and $(\bar{g}_{j,l})_{l\neq j}$.

Study of related system of variational inequalities

The main system : hypotheses

Hypothesis on the families of penalty costs
For any loop in Γ, any $(i_1, j_1), ..., (i_N, j_N)$ of Γ such that $(i_N, j_N) = (i_1, j_1)$, $\text{card}\{(i_1, j_1), ..., (i_N, j_N)\} = N - 1$ and $\forall\ q = 1, ..., N-1$, either $i_{q+1} = i_q$ or $j_{q+1} = j_q$, then $\forall (t, x)$,

$$\sum_{q=1,N-1} \varphi_{i_q, i_{q+1}}(t, x) \neq 0, \tag{4}$$

where either $\forall\ q = 1, ..., N-1$,
$\varphi_{i_q,i_{q+1}}(t,x) = -\underline{g}_{i_q,i_{q+1}}(t,x)\mathbf{1}_{i_q \neq i_{q+1}} + \bar{g}_{j_q,i_{q+1}}(t,x)\mathbf{1}_{j_q \neq j_{q+1}}$
or $\varphi_{i_q,i_{q+1}}(t,x) = \underline{g}_{i_q,i_{q+1}}(t,x)\mathbf{1}_{i_q \neq i_{q+1}} - \bar{g}_{j_q,i_{q+1}}(t,x)\mathbf{1}_{j_q \neq j_{q+1}}$.

Study of related system of variational inequalities

Notions of viscosity sub-supersolution of (3)

Definition :
$u = (u^{i,j})$: viscosity subsolution of (3) if u is *usc* and, if for $t < T$ and any (p_u, q_u, M_u) in $\bar{\mathcal{J}}^{+}(u^{i,j}(t,x))$,

$$\min\Big\{ \big(v^{i,j} - L^{i,j}[\vec{v}]\big)(t,x)\,;\ ;\ \max\Big\{\big(v^{i,j} - U^{i,j}[\vec{v}]\big)(t,x); \\ -p_u - q_u b(t,x) - \tfrac{1}{2}\mathrm{Tr}\left(\sigma\sigma^T M_u\right) - \Psi^{i,j}(t,x,(v^{k,l}(t,x)))\Big\}\Big\} \leq 0, \tag{5}$$

and $v^{i,j}(T,x) \leq h^{i,j}(x)$, for $t = T$.
$(v^{i,j})$: supersolution of (3) if v *lsc* and if (5) holds for any (p_v, q_v, M_v) in $\bar{\mathcal{J}}^{-}(v^{i,j}(t,x))$ replacing $\leq$ by $\geq$.

Study of related system of variational inequalities

The comparison result

The comparison result Assume that $u = (u^{i,j})$ (resp : $w = (w^{i,j})$) is a subsolution of (3) (is a supersolution of (3)), If, in addition both u and w are in class Π_g
$\exists\ C,\ \gamma > 0,\ \forall\ (t,x),\ |u^{i,j}(t,x)| + |w^{i,j}(t,x)| \leq C(1 + |x|^\gamma)$,
then

$$\forall t \in [0, T[,\ \forall\ (i,j)\ \ u^{i,j}(t,x) \leq w^{i,j}(t,x).$$

$\Rightarrow$ there exists at most one continuous viscosity solution in the class Π_g.

Auxiliary system of variational inequalities

For any $(i,j) \in \Gamma = \Gamma^1 \times \Gamma^2$ we introduce

$$\begin{cases} \max\Big\{ \big(v^{i,j} - U^{i,j}[\vec{v}]\big)(t,x)\,; \\ \min\Big\{\big(v^{i,j} - L^{i,j}[\vec{v}]\big)(t,x); \\ \qquad -\partial_t v^{i,j}(t,x) - \mathcal{L}v^{i,j}(t,x) - \Psi^{i,j}(t,x,(v^{k,l}(t,x)))\Big\}\Big\} = 0 \\ v^{i,j}(T,x) = h_{i,j}(x) \end{cases} \tag{6}$$

$$L^{i,j}[\vec{v}](t,x) := \max_{k\in(\Gamma^1)^{-i}} (v^{k,j}(t,x) - \underline{g}_{i,k}(t,x))$$

$$U^{i,j}[\vec{v}](t,x) = \min_{l\in(\Gamma^2)^{-j}} (v^{i,l}(t,x) + \bar{g}_{j,l}(t,x)).$$

Study of related system of variational inequalities

First approximating scheme

$\forall\ (i,j) \in \Gamma = \Gamma^1 \times \Gamma^2,$

$$\begin{cases} \min\{\bar{v}^{i,j,m}(t,x) - \max_{k \in (\Gamma^1)^{-i}} (\bar{v}^{k,j,m}(t,x) - \underline{g}_{i,k}(t,x)); \\ -\partial_t \bar{v}^{i,j,m}(t,x) - \mathcal{L}\bar{v}^{i,j,m}(t,x) - \bar{\Psi}^{i,j,m}(t,x,(\bar{v}^{k,l,m}(t,x)))\} = 0, \\ \bar{v}^{i,j,m}(T,x) = h^{i,j}(x) \end{cases} \tag{7}$$

$$L^{i,j,m}(\vec{\bar{v}}) = \max_{k \in (\Gamma^1)^{-i}} (\bar{v}^{k,j,m}(t,x) - \underline{g}_{i,k}(t,x))$$

$$U^{i,j,m}(\vec{\bar{v}}) = \min_{l \in (\Gamma^2)^{-j}} (v^{i,l,m}(s,x) + \bar{g}_{j,l}(s,x)$$

$$\bar{\Psi}^{i,j,m}(t,x,(y^{k,l})) = \Psi^{i,j}(t,x,(y^{k,l})) - m\big(y^{i,j} - \min_{l \in (\Gamma^2)^{-j}} (y^{i,l} + \bar{g}_{j,l}(t,x))\big)^{+}.$$

Study of related system of variational inequalities

Second approximating scheme

$\forall \ (i,j) \in \Gamma = \Gamma^1 \times \Gamma^2$,

$$\begin{cases} \max\{\underline{v}^{i,j,n}(t,x) - \min_{l \in (\Gamma^2)^{-j}} (\underline{v}^{i,l,n}(t,x) + \bar{g}_{j,l}(t,x)); \\ -\partial_t \underline{v}^{i,j,n}(t,x) - \mathcal{L}\underline{v}^{i,j,n}(t,x) - \underline{\Psi}^{i,j,n}(t,x,(\underline{v}^{k,l,n}(t,x)))\} = 0, \\ \underline{v}^{i,j,n}(T,x) = h^{i,j}(x) \end{cases} \tag{8}$$

with

$$\underline{\Psi}^{i,j,n}(t,x,(y^{k,l})) = \Psi^{i,j}(t,x,y^{i,j}) + n\big(\max_{k \in (\Gamma^1)^{-i}} (y^{k,j} - \underline{g}_{i,k}(t,x)) - y^{i,j}\big)^{+}$$

Identification of the limit of the two schemes

Theorem : viscosity characterization of the limit

• For each m, $(\bar{v}^{i,j,m})_{i,j}$: value of some standard switching problem,
$\lim_m \searrow \bar{v}^{i,j,m} = \bar{v}^{i,j}$, with $\bar{v}^{i,j}$: is *usc* and a (viscosity) solution to system (3).

• For each n $(\underline{v}^{i,j,n})$ coincides (up to a sign) with value of standard switching problem.
$\lim \nearrow \underline{v}^{i,j,n} = \underline{v}^{i,j}$ with $\underline{v}^{i,j}$ *lsc* and a (viscosity) solution to system (6).

Perron's method : existence of viscosity solution for systems (3) and (6)

Theorem

Suppose that system (3) satisfies the comparison theorem. If besides there exist both $\underline{v} = (\underline{v}^{i,j})$ *which is lsc and a supersolution of (3) and* $\bar{v}$ *which is usc and a subsolution of (3) then*

$$\exists\ u = (u^{i,j})\ s.t.\ \bar{v}^{i,j} \leq u^{i,j} \leq \underline{v}^{i,j},$$

with u which is continuous and a viscosity solution of (3).

Identification of the limit of the two schemes

Sketches of the proofs

- First claim : $\bar{v}^{i,j}$ viscosity solution of (3)
  - Step 1 : Prove that $\bar{v}^{i,j}$: subsolution of (3) and, for each m_0, v^{i,j,m_0} : supersolution
  - Step 2 : Set $v^{i,j,(m_0)} :=$ $\sup\{\tilde{v}^{i,j} \text{ subsolution s.t. } \bar{v}^{i,j} \leq \tilde{v}^{i,j} \leq v^{i,j,m_0}\}$
  - Step 3 : By uniqueness of viscosity solution, we get $v^{i,j} = \bar{v}^{i,j}$
- Second claim : $\underline{v}^{i,j}$ viscosity solution of (6).
  Main idea : replace $\underline{v}$ by $-\underline{v}$, verify that $-\underline{v}$ satisfies a new system of the same type as (3) and mimic the previous argumentation.

Third part : the switching game

Min-max and Max-min PDEs and connection with zero-sum Dynkin games

- Let consider a Brownian setting (finite horizon T) + X strong solution of

$$dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dW_s, \ \forall \ s \in [t, T]$$

  and $\mathcal{L}$ its infinitesimal generator
- $l(t,x)$, $h(t,x)$ and $g(x)$ continuous functions of Π^g such that

$$l(t,x) \leq h(t,x) \text{ and } l(T,x) \leq g(x) \leq h(T,x)$$

- $f(t,x,y,z)$ $\mathbb{R}$-valued function, Lipschitz in (y,z), in Π^g and continuous in (t,x) (uniformly w.r.t (y,z)).

Min-max and Max-min PDEs and connection with zero-sum Dynkin games

Let us now consider the following PDE with bilateral obstacles

$$\min\{(u-l)(t,x),\ \max\{(u-h)(t,x),\ -\partial_t u-\mathcal{L}u-f(t,x,u,(\sigma^T D_x u))\}\} \quad (9)$$

Theorem (Hamadene-Hassani 05)

There exists $u := u(t,x)$ a continuous function of the class Π^g which is the unique viscosity solution of system (9).
Besides $u(t,x)$ is also solution of

$$\max\{(u-h)(t,x),\ \min\{(u-l)(t,x),\ -\partial_t u-\mathcal{L}u-f(t,x,u,(\sigma^T D_x u))\}\} \quad (10)$$

The switching game : existence of the value of the game

Theorem

Assuming that

(i) The generators $\Psi^{i,j}$ do not depend on z and satisfies

$$\forall\,(s,x,\vec{y})\ \ |\Psi^{i,j}(s,x,\vec{y})| \leq C(1+|x|^{\gamma}).$$

(ii) the family $(\bar{g}_{j,l})$ of penalty costs are Itô processes, i.e.
$d\bar{g}_{j,l}(s) = \bar{u}_s^{j,l} ds + \bar{v}_s^{j,l} dW_s$, with $\bar{u}^{j,l}$ and $\bar{v}^{j,l}$ s.t.
$\mathbb{E}\left(\int_0^T |\bar{u}_s^{j,l}|^2 ds\right) < \infty$, and $\mathbb{E}\left(\int_0^T |\bar{v}_s^{j,l}|^2 ds\right) < \infty$,

• the two obstacles associated with $\bar{v}^{i,j}$ of system (3) are separated i.e. $L^{i,j}(\bar{v}) \leq U^{i,j}(\bar{v})$

• The two solutions $(\bar{v}^{i,j})$ and $(\underline{v}^{i,j})$ associated with systems (3) and (6) coincide.

Theorem

Under the additional assumption that the generator $\Psi^{i,j}$ (modelizing instantaneous profit in mode (i,j)) does not depend on $(\vec{y}, z)$ we also claim that

$$\bar{v}^{i,j} = \underline{v}^{i,j} = V^{i,j},$$

with

$$V^{i,j} = \text{ess inf}_{\beta \in \mathcal{B}_{t,j}} \text{ess sup}_{\alpha \in \mathcal{A}_{t,i}} J(\alpha, \beta) = \text{ess sup}_{\alpha \in \mathcal{A}_{t,i}} \text{ess inf}_{\beta \in \mathcal{B}_{t,j}} J(\alpha, \beta)$$

which is the value of the switching game.

Thanks for your attention !