Stochastic target games

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Joint works with L. Moreau (ETH-Zürich) and M. Nutz (Columbia)

Problem formulation and Motivations

Problem formulation

Provide a PDE characterization of the *viability* sets

$$\Lambda(t) := \{(z,m) : \exists\, \mathfrak{u} \in \mathfrak{U} \text{ s. t. } \mathbb{E}\left[\ell(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(T))|\mathcal{F}_t\right] \geq m \;\forall\, \vartheta \in \mathcal{V}\}$$

In which :

- $\mathcal{V}$ is a set of admissible adverse controls
- $\mathfrak{U}$ is a set of admissible strategies
- $Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}$ is an adapted $\mathbb{R}^d$-valued process s.t. $Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(t) = z$
- ℓ is a given loss/utility function
- m a threshold.

Application in finance

- $Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta} = (X_{t,x}^{\mathfrak{u}[\vartheta],\vartheta}, Y_{t,x,y}^{\mathfrak{u}[\vartheta],\vartheta})$ where
  - $X_{t,x}^{\mathfrak{u}[\vartheta],\vartheta}$ models financial assets or factors with dynamics depending on ϑ
  - $Y_{t,x,y}^{\mathfrak{u}[\vartheta],\vartheta}$ models a wealth process
  - ϑ is the control of the market : parameter uncertainty (e.g. volatility), adverse players, etc...
  - $\mathfrak{u}[\vartheta]$ is the financial strategy given the past observations of ϑ.

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Robust partial hedging under uncertainty and related price :

$$\inf\{y : \exists\, \mathfrak{u} \text{ s.t. } \mathbb{E}\left[\Psi\left(Y_{t,x,y}^{\mathfrak{u}[\vartheta],\vartheta}(T) \geq g(X_{t,x}^{\mathfrak{u}[\vartheta],\vartheta}(T))\right)\right] \geq m \;\forall\, \vartheta\}$$

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Robust hedging under uncertainty and related price :

$$\inf\{y : \exists\, \mathfrak{u} \text{ s.t. } Y_{t,x,y}^{\mathfrak{u}[\vartheta],\vartheta}(T) \geq g(X_{t,x}^{\mathfrak{u}[\vartheta],\vartheta}(T))\ \forall\, \vartheta\}$$

□ Flexible enough to embed constraints, transaction costs, market impact, etc...

Setting for this talk

(see the papers for abstract versions)

Brownian diffusion setting

Brownian diffusion setting

□ **State process :** $Z^{\mathfrak{u}[\vartheta],\vartheta}$ solves (μ and σ continuous, uniformly Lipschitz in space)

$$Z(s) = z + \int_t^s \mu(Z(r), \mathfrak{u}[\vartheta]_r, \vartheta_r)\, dr + \int_t^s \sigma(Z(r), \mathfrak{u}[\vartheta]_r, \vartheta_r)\, dW_r$$

□ The loss function ℓ has polynomial growth and is continuous.

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□ The loss function ℓ has polynomial growth and is continuous.

□ **Controls and strategies :**

- $\mathcal{V}$ is the set of predictable processes with values in $V \subset \mathbb{R}^d$.
- $\mathfrak{U}$ is set of non-anticipating maps $\mathfrak{u} : \vartheta \in \mathcal{V} \mapsto \mathcal{U}$, i.e.

$$\{\omega : \vartheta_1(\omega) =_{[0,s]} \vartheta_2(\omega)\} \subset \{\omega : \mathfrak{u}[\vartheta_1](\omega) =_{[0,s]} \mathfrak{u}[\vartheta_2](\omega)\}.$$

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The game problem

- **The *viability* sets** are given by

$$\Lambda(t) \quad := \quad \{(z,m) : \exists\, \mathfrak{u} \in \mathfrak{U} \text{ s. t. } \mathbb{E}\left[\ell(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(T))|\mathcal{F}_t\right] \geq m \;\forall\, \vartheta \in \mathcal{V}\}$$

Compare with the formulation of games in Buckdahn and Li (08).

Geometric dynamic programming principle for controlled loss cases

How are the properties

$(z, m) \in \Lambda(t)$ and $(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(\theta), ?) \in \Lambda(\theta)$

related ?

$$\Lambda(t) \quad := \quad \{(z,m) : \exists\, \mathfrak{u} \in \mathfrak{U} \text{ s. t. } \mathbb{E}\left[\ell(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(T))|\mathcal{F}_t\right] \geq m \;\forall\, \vartheta \in \mathcal{V}\}$$

Unformal derivation

- Take $(z, m) \in \Lambda(t)$ and $\mathfrak{u} \in \mathfrak{U}$ such that

$$\operatorname*{ess\,inf}_{\vartheta \in \mathcal{V}} \mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(T)\right)|\mathcal{F}_t\right] \geq m \;\; \mathbb{P} - \text{a.s.}$$

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Take care of the evolution of the worst case scenario conditional expectation :

$$S_r^{\vartheta} := \operatorname*{ess\,inf}_{\bar{\vartheta} \in \mathcal{V}} \mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u}[\vartheta \oplus_r \bar{\vartheta}],\vartheta \oplus_r \bar{\vartheta}}(T)\right) |\mathcal{F}_r\right],$$

where $\vartheta \oplus_r \bar{\vartheta} = \vartheta \mathbf{1}_{[0,r]} + \mathbf{1}_{(r,T]}\bar{\vartheta}$.

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where $\vartheta \oplus_r \bar{\vartheta} = \vartheta \mathbf{1}_{[0,r]} + \mathbf{1}_{(r,T]}\bar{\vartheta}$.

Then

$$S^{\vartheta} \text{ is a submartingale and } S_t^{\vartheta} \geq m \text{ for all } \vartheta \in \mathcal{V},$$

and we can find a martingale M^{ϑ} such that

$$S^{\vartheta} \geq M^{\vartheta} \text{ and } M_t^{\vartheta} = S_t^{\vartheta} \geq m.$$

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□ Take $(z, m) \in \Lambda(t)$ and $\mathfrak{u} \in \mathfrak{U}$ such that

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where $\vartheta \oplus_r \bar{\vartheta} = \vartheta \mathbf{1}_{[0,r]} + \mathbf{1}_{(r,T]}\bar{\vartheta}$.

Hence,

$$\operatorname*{ess\,inf}_{\bar{\vartheta} \in \mathcal{V}} \mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u}[\vartheta \oplus_\theta \bar{\vartheta}],\vartheta \oplus_\theta \bar{\vartheta}}(T)\right)|\mathcal{F}_\theta\right] = S_\theta^{\vartheta} \geq M_\theta^{\vartheta} \ \ \mathbb{P}-\text{a.s.}$$

and therefore there exists a martingale M^{ϑ} such that $M_t^{\vartheta} = m$ and

$$\left(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(\theta), M_\theta^{\vartheta}\right) \in \Lambda(\theta) \ \mathbb{P}-\text{a.s.}$$

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and therefore there exists a predictable $\alpha^{\vartheta} \in \mathcal{A}$ such that

$$(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(\theta), M_{t,m}^{\alpha^{\vartheta}}(\theta)) \in \Lambda(\theta) \; \mathbb{P}-\text{a.s.} \;, \;\; M_{t,m}^{\alpha^{\vartheta}} := m + \int_t^{\cdot} \alpha_s^{\vartheta} dW_s$$

The geometric dynamic programming principle

(GDP1) : If $(z, m) \in \Lambda(t)$, then $\exists\ \mathfrak{u} \in \mathfrak{U}$ and $\{\alpha^{\vartheta}, \vartheta \in \mathcal{V}\} \subset \mathcal{A}$ such that

$$(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(\theta), M_{t,m}^{\alpha^{\vartheta}}(\theta)) \in \Lambda(\theta)\ \mathbb{P} - \text{a.s.}\ \forall\ \vartheta \in \mathcal{V}.$$

(GDP2) : If $(\mathfrak{u}, \mathfrak{a}) \in \mathfrak{U} \times \mathfrak{A}$ are such that

$$(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(\theta[\vartheta]), M_{t,m}^{\mathfrak{a}[\vartheta]}(\theta[\vartheta])) \in \Lambda(\theta[\vartheta])\ \mathbb{P} - \text{a.s.}\ \forall\ \vartheta \in \mathcal{V}$$

for some family $(\theta[\vartheta], \vartheta \in \mathcal{V})$ of non-anticipating stopping times, then

$$(z, m) \in \Lambda(t).$$

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for some family $(\theta[\vartheta], \vartheta \in \mathcal{V})$ of non-anticipating stopping times, then

$$(z,m) \in \Lambda(t).$$

Rem : Use heavily the regularity of the constraint in expectation (ℓ continuous + unif. Lipschitz coefficients). Exact statement requires an extra relaxation, which does not alter the pde derivation. See Bouchard, Moreau and Nutz, AAP to appear.

PDE Characterization

□ **Monotone case :** $Z_{t,x,y}^{\mathfrak{u}[\vartheta],\vartheta} = (X_{t,x}^{\mathfrak{u}[\vartheta],\vartheta}, Y_{t,x,y}^{\mathfrak{u}[\vartheta],\vartheta})$ with values in $\mathbb{R}^d \times \mathbb{R}$ with $X_{t,x}^{\mathfrak{u}[\vartheta],\vartheta}$ independent of y and $\ell \uparrow y$.

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□ **The value function is :**

$$\varpi(t,x,m) := \inf\{y : (x,y,m) \in \Lambda(t)\}.$$

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□ We have the "characterization"

$$y > \varpi(t,x,m) \Rightarrow (z,m) \in \Lambda(t) \Rightarrow y \geq \varpi(t,x,m)$$

PDE characterization - "waving hands" version

- Assuming smoothness, existence of optimal strategies...

- $y = \varpi(t, x, m)$ implies
$Y^{\mathfrak{u}[\vartheta],\vartheta}(t+) \geq \varpi(t+, X^{\mathfrak{u}[\vartheta],\vartheta}(t+), M^{\mathfrak{a}[\vartheta]}(t+))$ for all ϑ.

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This implies $dY^{\mathfrak{u}[\vartheta],\vartheta}(t) \geq d\varpi(t, X^{\mathfrak{u}[\vartheta],\vartheta}(t), M^{\mathfrak{a}[\vartheta]}(t))$ for all ϑ

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Hence, for all ϑ,

$$\begin{aligned}
\mu_Y(x,y,\mathfrak{u}[\vartheta]_t,\vartheta_t) &\geq \mathcal{L}_{X,M}^{\mathfrak{u}[\vartheta]_t,\vartheta_t,\mathfrak{a}[\vartheta]_t}\varpi(t,x,m) \\
\sigma_Y(x,y,\mathfrak{u}[\vartheta]_t,\vartheta_t) &= \sigma_X(x,\mathfrak{u}[\vartheta]_t,\vartheta_t)D_x\varpi(t,x,m) \\
&\quad +\mathfrak{a}[\vartheta]_t D_m\varpi(t,x,m)
\end{aligned}$$

with $y = \varpi(t,x,m)$

PDE characterization - "waving hands" version

□

$$\sup_{(u,a)\in\mathcal{N}^v\varpi}\left(\mu_Y(\cdot,\varpi,u,v)-\mathcal{L}^{u,v,a}_{X,M}\varpi\right)\geq 0$$

where

$$\mathcal{N}^v\varpi := \{(u,a)\in U\times\mathbb{R}^d : \sigma_Y(\cdot,\varpi,u,v)=\sigma_X(\cdot,u,v)D_x\varpi + aD_m\varpi\}.$$

PDE characterization - "waving hands" version

□

$$\inf_{v\in V} \sup_{(u,a)\in\mathcal{N}^v\varpi} \left(\mu_Y(\cdot,\varpi,u,v) - \mathcal{L}^{u,v,a}_{X,M}\varpi\right) \geq 0$$

where

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PDE characterization - "waving hands" version

- **Supersolution property**

$$\inf_{v \in V} \sup_{(u,a) \in \mathcal{N}^v \varpi} \left(\mu_Y(\cdot, \varpi, u, v) - \mathcal{L}^{u,v,a}_{X,M} \varpi \right) \geq 0$$

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Supersolution property

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Subsolution property

$$\sup_{(u[\cdot],a[\cdot])\in\mathcal{N}^{[\cdot]}\varpi} \inf_{v\in V} \left(\mu_Y(\cdot,\varpi,u[v],v) - \mathcal{L}^{u[v],v,a[v]}_{X,M}\varpi\right) \leq 0$$

where

$$\mathcal{N}^{[\cdot]}\varpi := \{\text{loc. Lip. } (u[\cdot],a[\cdot]) \text{ s.t. } (u[\cdot],a[\cdot])\in\mathcal{N}^{\cdot}\varpi(\cdot)\}.$$

Geometric dynamic programming principle for the a.s. constraint case

Prove that

$$y \geq \varpi(t,x) \Leftrightarrow Y_{t,x,y}^{\mathfrak{u}[\vartheta],\vartheta}(\theta) \geq \varpi(\theta, X_{t,x}^{\vartheta}(\theta))$$

(when X does not depend on $\mathfrak{u}$)

$$\varpi(t,x) := \inf\{y : \exists\, \mathfrak{u} \in \mathfrak{U} \text{ s. t. } Y_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(T) \geq g(X_{t,x}^{\vartheta}(T)) \; a.s. \; \forall\, \vartheta \in \mathcal{V}\}$$

Geometric dynamic programming principle for the a.s. constraint case

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Expected loss case : play with the regularity of the constraint in expectation form. Not possible here.

Geometric dynamic programming principle for the a.s. constraint case

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No adverse control case : measurable selection argument which requires to have a Polish space structure. Not possible here.

Geometric dynamic programming principle for the a.s. constraint case

Prove that

$$y \geq \varpi(t,x) \Leftrightarrow Y^{\mathfrak{u}[\vartheta],\vartheta}_{t,x,y}(\theta) \geq \varpi(\theta, X^{\vartheta}_{t,x}(\theta))$$

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Main difficulty : no smoothness and no measurable selection argument possible.

GDP1 - “Easy part”

□ **GDP1** Assume that $y > \varpi(t,x)$. Then, there exists $\mathfrak{u} \in \mathfrak{U}$ such that

$$Y^{\mathfrak{u},\vartheta}_{t,x,y}(\theta) \geq \varpi_*(\theta, X^{\vartheta}_{t,x}(\theta)) \ a.e. \ \ \forall \, \vartheta \in \mathcal{V}.$$

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This implies as before that ϖ_* is a supersolution of

$$H\varpi_* := \inf_{v \in V} \sup_{u \in \mathcal{N}^v \varpi_*} \left(\mu_Y(\cdot, \varpi_*, u, v) - \mathcal{L}_X^v \varpi_*\right) \geq 0$$

where

$$\mathcal{N}^v \varpi_* := \{u \in U : \sigma_Y(\cdot, \varpi_*, u, v) = \sigma_X(\cdot, v) D\varpi_*\}.$$

GDP2 - "Difficult part"

□ Assume : $\forall$ compact B and $\eta > 0$, $\exists$ a smooth w s.t.

$w \leq \varpi + \eta$ on B , $Hw \geq 0$ on $[0, T) \times \mathbb{R}^d$ and $w \geq g$ on $\{T\} \times \mathbb{R}^d$

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- Assume : $\exists !\ \hat{u} = \hat{u}(x, y, \rho, v)$ s.t. $\sigma_Y(x, y, \hat{u}, v) = \rho$.

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☐ Assume : $\exists !\ \hat{u} = \hat{u}(x, y, \rho, v)$ s.t. $\sigma_Y(x, y, \hat{u}, v) = \rho$.

Then, use a verification (under regularity assumptions) ensuring that

$$Y^{\mathfrak{u}_o,\vartheta}_{t,x,y}(\theta) \geq w(\theta, X^{\vartheta}_{t,x}(\theta)) \ \ a.e. \ \forall \, \vartheta \in \mathcal{V}$$

implies that the Markovian strategy

$$\bar{\mathfrak{u}}[\vartheta] := \mathfrak{u}_o \mathbf{1}_{[t,\theta)} + 1_{[\theta,T]} \hat{u}(Z^{\bar{\mathfrak{u}},\vartheta}_{t,x,y}, [\sigma_X(\cdot, \vartheta) D_X w](\cdot, X^{\vartheta}_{t,x}), \vartheta)$$

is such that

$$Y^{\bar{\mathfrak{u}},\vartheta}_{t,x,y}(T) \geq g(X^{\vartheta}_{t,x}(T)) \ \ a.e. \ \forall \, \vartheta \in \mathcal{V},$$

since

$$\mu_Y(\cdot, w, \hat{u}(\cdot, w, \sigma_X(\cdot, v) Dw, v), v) - \mathcal{L}^v_X w \geq 0 \ \text{ and } \ w(T, \cdot) \geq g.$$

GDP2 - "Difficult part"

□ **GDP2** Let ϕ be a test function for ϖ^* at (t,x). Let $\eta > 0$ be such that

$$Y_{t,x,y}^{\mathfrak{u}_o,\vartheta}(\theta) \geq \varpi(\theta, X_{t,x}^{\vartheta}(\theta)) + \eta \;\; a.e. \; \forall \; \vartheta \in \mathcal{V},$$

where θ is the first exit time from an open ball $\mathcal{O} \ni (t,x)$. Then, $y \geq \varpi(t,x)$.

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where θ is the first exit time from an open ball $\mathcal{O} \ni (t,x)$. Then, $y \geq \varpi(t,x)$.

□ Indeed, let w be the smooth supersolution constructed as above so that

$$\varpi + \eta \geq w \ \text{ on } \ \partial\mathcal{O}.$$

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□ **GDP2** Let ϕ be a test function for ϖ^* at (t,x). Let $\eta > 0$ be such that

$$Y^{\mathfrak{u}_o,\vartheta}_{t,x,y}(\theta) \geq \varpi(\theta, X^{\vartheta}_{t,x}(\theta)) + \eta \;\; a.e. \; \forall\, \vartheta \in \mathcal{V},$$

where θ is the first exit time from an open ball $\mathcal{O} \ni (t,x)$. Then, $y \geq \varpi(t,x)$.

□ Indeed, let w be the smooth supersolution constructed as above so that

$$\varpi + \eta \geq w \;\; \text{on} \;\; \partial\mathcal{O}.$$

Then,

$$Y^{\mathfrak{u}_o,\vartheta}_{t,x,y}(\theta) \geq w(\theta, X^{\vartheta}_{t,x}(\theta)) \;\; a.e. \; \forall\, \vartheta \in \mathcal{V}.$$

From this, we can use the Markovian strategy based on $\hat{u}$ to reach the target at T for all $\vartheta \in \mathcal{V}$.

GDP2 - "Difficult part"

- We assume now : $\exists\,!\ \hat{u} = \hat{u}(x, y, \rho, v)$ s.t. $\sigma_Y(x, y, \hat{u}, v) = \rho$.

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□ We need to prove that : $\forall$ compact B and $\eta > 0$, $\exists$ a smooth w s.t.

$$w \leq \varpi + \eta \quad \text{on } B\ , \quad Hw \geq 0 \quad \text{on} \quad [0, T) \times \mathbb{R}^d \quad \text{and } w \geq g \quad \text{on} \quad \{T\} \times \mathbb{R}^d$$

where

$$Hw := \inf_v \mu_Y(\cdot, w, \hat{u}(\cdot, w, \sigma_X(\cdot, v)Dw, v), v) - \mathcal{L}_X^v w$$

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☐ Assume that $(y, z) \mapsto \mu_Y(\cdot, y, \hat{u}(\cdot, y, z, v), v)$ can serve as a nice driver for BSDEs. Then, the above is the HJB equation of an optimal control on BSDEs.

GDP2 - "Difficult part"

- Then, it is the same for the version with *shaken coefficients*

$$H^{\varepsilon}\varphi := \inf_{v, \|b\| \leq 1} \mu_Y(\varepsilon b + \cdot, \varphi, \hat{u}(\varepsilon b + \cdot, \varphi, \sigma_X(\varepsilon b + \cdot, v) D\varphi, v), v) - \mathcal{L}_X^{v, \varepsilon b}\varphi$$

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☐ Using stability results, we show that for each $\varepsilon > 0$ $\exists$ viscosity supersolution w^ε of (a version of)

$$H^\varepsilon w^\varepsilon \ge 0 \quad \text{on} \quad [0, T) \times \mathbb{R}^d \quad \text{and } w^\varepsilon \ge g \quad \text{on} \quad \{T\} \times \mathbb{R}^d$$

s.t. $w^\varepsilon \le \varpi + \eta$ on B for $\varepsilon > 0$ small.

GDP2 - "Difficult part"

□ Then, it is the same for the version with *shaken coefficients*

$$H^{\varepsilon}\varphi := \inf_{v,\|b\|\leq 1} \mu_Y(\varepsilon b+\cdot, \varphi, \hat{u}(\varepsilon b+\cdot, \varphi, \sigma_X(\varepsilon b+\cdot, v)D\varphi, v), v) - \mathcal{L}_X^{v,\varepsilon b}\varphi$$

□ Using stability results, we show that for each $\varepsilon > 0$ $\exists$ viscosity supersolution w^{ε} of (a version of)

$$H^{\varepsilon}w^{\varepsilon} \geq 0 \ \text{ on } \ [0,T)\times\mathbb{R}^d \ \text{ and } w^{\varepsilon} \geq g \ \text{ on } \ \{T\}\times\mathbb{R}^d$$

s.t. $w^{\varepsilon} \leq \varpi + \eta$ on B for $\varepsilon > 0$ small.

□ Finally, assuming that $H(x,\cdot)$ is concave and using the fact that on $B_{\varepsilon/2}(t,x)$

$$\inf_v \mu_Y^{t,x}(w^{\varepsilon}(\cdot), \hat{u}^{t,x}(w^{\varepsilon}(\cdot), \sigma_X^{t,x}(v)Dw^{\varepsilon}(\cdot), v), v) - \mathcal{L}_X^{v,(t,x)}w^{\varepsilon}(\cdot) \geq 0,$$

we can smooth w^{ε} to obtain w.

Application to hedging under uncertainty

- Evolution of the log price :

$$X^{\vartheta}_{t,x} = x + \int_t^{\cdot} \mu(s, X^{\vartheta}_{t,x}(s), \vartheta_s)ds + \int_t^{\cdot} \sigma(s, X^{\vartheta}_{t,x}(s), \vartheta_s)dW_s.$$

Application to hedging under uncertainty

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□ Financial strategy (with different interest rates for borrowing and lending) :

$$\begin{aligned} Y^{\nu,\vartheta}_{t,x,y} &= y + \int_t^{\cdot} \left(\nu_s^\top \{\mu + \tfrac{1}{2}\gamma\}(s, X^{\vartheta}_{t,x}(s), \vartheta_s) + \rho(s, Y^{\nu,\vartheta}_{t,x,y}(s), \nu_s, \vartheta_s) \right) \\ &\quad + \int_t^{\cdot} \nu_s^\top \sigma(s, X^{\vartheta}_{t,x}(s), \vartheta_s) dW_s, \end{aligned}$$

where $\gamma =$ vector of diagonal elements of $\sigma\sigma^\top$ and

$$\rho(t, y, u, v) := [y - u^\top \mathbf{1}]^+ r^l(t, v) - [y - u^\top \mathbf{1}]^- r^b(t, v).$$

Application to hedging under uncertainty

□ The super-hedging price under uncertainty

$$\inf\{y:\ \exists\, \mathfrak{u} \in \mathfrak{U}\ \text{ s.t. }\ Y^{\mathfrak{u},\vartheta}_{t,x,y}(T) \geq g(X^{\vartheta}_{t,x}(T))\ \mathbb{P}-\text{a.s.}\ \forall\, \vartheta \in \mathcal{V}\}$$

is a (discontinuous) viscosity solution of

$$\inf_{v \in V} \left[\rho(\cdot,\varphi,D\varphi,v) + \tfrac{1}{2}\gamma(\cdot,v)D\varphi - \mathcal{L}^{v}_{X}\varphi\right] = 0 \quad \text{on} \quad [0,T)\times\mathbb{R}^d$$
$$\varphi(T,\cdot) = g \quad \text{on} \quad \mathbb{R}^d.$$

We retrieve a version of the Black-Scholes-Barrenblatt equation.

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