Ergodic BSDEs related to PDEs with Neumann boundary conditions in a weakly dissipative environment

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Recalls on BSDEs

General BSDE

$$Y_t = \xi + \int_t^T \psi(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \qquad \forall t \in [0, T]$$

where a solution is a couple (Y, Z) such that :

- Y and Z are progressively measurable
- $\mathbb{P}$-a.s. :

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \qquad 0 \leq t \leq T$$

Recalls on BSDEs : the monotonic case

$$Y_t = Y_T + \int_t^T \psi(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \qquad \forall 0 \leq t \leq T < +\infty$$

Hypothesis :

- ψ is M_ψ-Lipschitz in y and z
- ψ is monotonic in y : $\exists \mu > 0$, $\forall y_1, y_2 \in \mathbb{R}$,

$$(\psi(s, y_1, z) - \psi(s, y_2, z)).(y_1 - y_2) \leq -\mu |y_1 - y_2|^2$$

- $|\psi(t, 0, 0)| \leq K$

Theorem (Briand-Hu 1998)

There exists a unique solution (Y, Z) to the BSDE such that Y is bounded adapted continuous process, $\mathbb{E} \int_0^T |Z_s|^2 \mathrm{d}s < +\infty$, i.e. $Z \in \mathscr{M}^2$,

$$|Y_t| \leq \frac{M_\psi}{\mu}.$$

Recalls on EBSDEs

What is an EBSDE ?

$$Y_t = Y_T + \int_t^T [\psi(X_s, Z_s) - \lambda]\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \quad \forall 0 \leq t \leq T < +\infty$$

Solution

$\Rightarrow$ triplet (Y, Z, λ)

$$X_t^x = x + \int_0^t AX_s^x + F(X_s^x)\mathrm{d}s + \int_0^t G\mathrm{d}W_s, \qquad t \geq 0;$$
$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda]\mathrm{d}s - \int_t^T Z_s^x \mathrm{d}W_s, \qquad 0 \leq t \leq T + \infty.$$

Fuhrman, Hu, Tessitore (2007)

- **strict** dissipative assumption, $\exists \eta > 0$, $\forall x, y$,
  $< Ax + F(x) - Ay - F(y), x - y > \quad \leq \quad -\eta|x - y|^2$

$\Rightarrow$ key result : $|X_t^x - X_t^y| \leq e^{-\eta t}|x - y|$

Recalls on EBSDEs : Infinite dimensional framework

$$X_t^x = x + \int_0^t AX_s^x + F(X_s^x)\mathrm{d}s + \int_0^t G\,\mathrm{d}W_s, \qquad t \geq 0;$$

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda]\mathrm{d}s - \int_t^T Z_s^x \mathrm{d}W_s, \qquad 0 \leq t \leq T + \infty.$$

Debussche, Hu, Tessitore (2010)

- **weak** dissipative assumption :
  $\exists \eta > 0, \forall x, y, \quad < Ax - Ay, x - y > \leq -\eta |x - y|^2$
  F is Lipschitz bounded and Gâteaux differentiable

$\Rightarrow$ key result Basic Coupling Estimate : $\forall \phi \in B_b$,
$|\mathscr{P}_t[\phi](x) - \mathscr{P}_t[\phi](y)| \leq C(1 + |x|^2 + |y|^2)e^{-\hat{\eta}t}$
where $\mathscr{P}_t[\Phi](x) = \mathbb{E}\Phi(X_t^x)$.

Recalls on EBSDEs : Finite dimensional framework

$G = \{\phi > 0\}$ a convex subset of $\mathbb{R}^d$ where lives the reflected the process X_t^x.

$$\begin{cases} X_t^x = x + \int_0^t f(X_s^x)\mathrm{d}s + \int_0^t \sigma(X_s^x)\mathrm{d}W_s + \int_0^t \nabla\phi(X_s^x)\mathrm{d}K_s^x, \quad t \geq 0; \\ K_t^x = \int_0^t \mathbb{1}_{\{X_s^x \in \partial G\}}\mathrm{d}K_s^x, \quad K_\cdot^x \text{ is non decreasing;} \end{cases}$$

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda]\mathrm{d}s + \int_t^T [g(X_s^x) - \mu]\mathrm{d}K_s^x - \int_t^T Z_s^x \mathrm{d}W_s, \quad 0 \leq t \leq T + \infty.$$

2 kinds of solutions

- (Y, Z, λ) if μ is given ;
- (Y, Z, μ) is λ is given.

Recalls on EBSDEs

$$\begin{cases} X_t^x = x + \int_0^t f(X_s^x)\mathrm{d}s + \int_0^t \sigma(X_s^x)\mathrm{d}W_s + \int_0^t \nabla\phi(X_s^x)\mathrm{d}K_s^x, \quad t \geq 0; \\ \\ K_t^x = \int_0^t \mathbb{1}_{\{X_s^x \in \partial G\}}\mathrm{d}K_s^x, \quad K_\cdot^x \text{ is non decreasing;} \end{cases}$$

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda]\mathrm{d}s + \int_t^T [g(X_s^x) - \mu]\mathrm{d}K_s^x$$
$$- \int_t^T Z_s^x \mathrm{d}W_s, \qquad 0 \leq t \leq T + \infty.$$

Richou (2009)

- G a bounded convex subset of $\mathbb{R}^d$;
- X_t^x lives in G ;
- **strict** dissipative assumption, $\exists \eta > 0$, $\forall x, y$,

$$< f(x) - f(y), x - y > + \frac{1}{2}\mathrm{Tr}[(\sigma(x) - \sigma(y))^t(\sigma(x) - \sigma(y))] \leq -\eta|x - y|^2.$$

$$-\eta + K_{\psi,z}K_\sigma \quad < \quad 0.$$

$\Rightarrow$ key result : $|X_t^x - X_t^y| \leq e^{-\eta t}|x - y|$ (when σ is constant)

EBDSE : our framework

$$\begin{cases} X_t^x = x + \int_0^t f(X_s^x)\mathrm{d}s + \int_0^t \sigma(X_s^x)\mathrm{d}W_s + \int_0^t \nabla\phi(X_s^x)\mathrm{d}K_s^x, \quad t \geq 0; \\ K_t^x = \int_0^t \mathbb{1}_{\{X_s^x \in \partial G\}}\mathrm{d}K_s^x, \quad K_\cdot^x \text{ is non decreasing;} \end{cases}$$

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda]\mathrm{d}s + \int_t^T [g(X_s^x) - \mu]\mathrm{d}K_s^x - \int_t^T Z_s^x \mathrm{d}W_s, \quad 0 \leq t \leq T + \infty.$$

Our framework

- G a convex subset of $\mathbb{R}^d$;
- $(X_t^x)_{t\geq 0}$ lives in G ;
- weak dissipative assumptions, $f = d + b$, :
  - d is dissipative : $\exists \eta > 0$, $\forall x, y$, $(d(x) - d(y)).(x - y) \leq -\eta|x - y|^2$;
  - b is bounded Lipschitz by B ;
- d is locally Lipschitz ;
- σ is invertible, both σ and σ^{-1} are bounded.

EBSDE : Existence

Initial problem : zero Neumann boundary conditions

$$\begin{cases} X_t^x = x + \int_0^t (d+b)(X_s^x)\mathrm{d}s + \int_0^t \sigma(X_s^x)\mathrm{d}W_s + \int_0^t \nabla\phi(X_s^x)\mathrm{d}K_s^x, \quad t \geq 0; \\ K_t^x = \int_0^t \mathbb{1}_{\{X_s^x \in \partial G\}}\mathrm{d}K_s^x, \quad K_\cdot^x \text{ is non decreasing;} \end{cases}$$

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda]\mathrm{d}s - \int_t^T Z_s^x \mathrm{d}W_s, \qquad 0 \leq t \leq T + \infty.$$

turns into

Penalized problem

$$X_t^{x,n} = x + \int_0^t (d + F_n + b)(X_s^{x,n})\mathrm{d}s + \int_0^t \sigma(X_s^{x,n})\mathrm{d}W_s.$$

$$Y_t^{x,n} = Y_T^{x,n} + \int_t^T [\psi(X_s^{x,n}, Z_s^{x,n}) - \lambda^n]\mathrm{d}s$$

$$- \int_t^T Z_s^{x,n}\mathrm{d}W_s, \quad 0 \leq t \leq T < +\infty.$$

Penalization method

Term of penalization

$$F_n(x) = -2n(x - \Pi(x)), \qquad \forall x \in \mathbb{R}^d$$

Key property of F_n

- F_n is 0-dissipative

$$X_t^{x,n} = x + \int_0^t (d + F_n + b)(X_s^{x,n})\mathrm{d}s + \int_0^t \sigma(X_s^{x,n})\mathrm{d}W_s$$

Basic coupling estimate

Lemma : Basic coupling estimate

There exists $C > 0$ and $\mu > 0$ such that $\forall \Phi \in B_b(\mathbb{R}^d)$,

$$|\mathscr{P}_t^n[\Phi](x) - \mathscr{P}_t^n[\Phi](y)| \leq C(1 + |x|^2 + |y|^2)e^{-\mu t}|\Phi|_0 \quad (1)$$

where $\mathscr{P}_t[\Phi](x) = \mathbb{E}\Phi(X_t^{x,n})$.

Classical approach : transform the EBSDE into a monotonic BSDE

Penalized problem

$$X_t^{x,n} = x + \int_0^t (d + F_n + b)(X_s^{x,n})\mathrm{d}s + \int_0^t \sigma(X_s^{x,n})\mathrm{d}W_s.$$

$$Y_t^{x,\alpha,n} = Y_T^{x,\alpha,n} + \int_t^T [\psi(X_s^{x,n}, Z_s^{x,\alpha,n}) - \alpha Y_s^{x,\alpha,n}]\mathrm{d}s - \int_t^T Z_s^{x,\alpha,n}\mathrm{d}W_s, \quad 0 \le t \le T < +\infty.$$

We define

$$v^{\alpha,n}(x) := Y_0^{x,\alpha,n}$$

Basic coupling estimates implies that :

$$|v^{\alpha,n}(x) - v^{\alpha,n}(y)| \le C(1 + |x|^2 + |y|^2)$$

Classical approach : fail

We need

$$|v^{\alpha,n}(x) - v^{\alpha,n}(y)| \leq C(1 + |x|^2 + |y|^2)|x - y|$$

$\Rightarrow$ Regularization.

Regularized penalized problem

$$X_t^{x,n,\varepsilon} = x + \int_0^t (d^\varepsilon + F_n^\varepsilon + b^\varepsilon)(X_s^{x,n,\varepsilon})\mathrm{d}s + \int_0^t \sigma^\varepsilon(X_s^{x,n,\varepsilon})\mathrm{d}W_s.$$

$$Y_t^{x,\alpha,n,\varepsilon} = Y_T^{x,\alpha,n,\varepsilon} + \int_t^T [\psi^\varepsilon(X_s{}^{x,n,\varepsilon}, Z_s^{x,\alpha,n,\varepsilon}) - \alpha Y_s^{x,\alpha,n,\varepsilon}]\mathrm{d}s - \int_t^T Z_s^{x,\alpha,n,\varepsilon}\mathrm{d}W_s, \quad 0 \leq t \leq T < +\infty$$

We define :

$$v^{\alpha,n,\varepsilon}(x) := Y_0^{\alpha,n,\varepsilon}$$

Construction of a solution

We want :

$$|\nabla v^{\alpha,n,\varepsilon}(x)| \leq C(1+|x|^2)$$

$\Rightarrow$ Additionnal assumptions :

- $|d(x)| \leq C(1+|x|^p)$
- $\exists c > 0$ such that ${}^t\xi\sigma(x){}^t\sigma(x)\xi \geq c|\xi|^2$

Then :

$$|v^{\alpha,n,\varepsilon}(x) - v^{\alpha,n,\varepsilon}(y)| \leq C(1+|x|^2+|y|^2)|x-y|$$

By a diagonale procedure :

$$(Y_t^{x,\alpha,n,\varepsilon} - Y_0^{x,\alpha,n,\varepsilon}, Z_t^{x,\alpha,n,\varepsilon}, \alpha v^{\alpha,n,\varepsilon}(0)) \underset{\alpha\to 0,\, n\to+\infty,\, \varepsilon\to 0}{\longrightarrow} (Y_t, Z_t, \lambda)$$

Main Result

Theorem

Assume that

- G a convex subset of $\mathbb{R}^d$;
- $(X_t^x)_{t\geq 0}$ is reflected in G ;
- f is locally Lipschitz ;
- weak dissipative assumptions, $f = d + b$, :
  - d is dissipative : $\exists \eta > 0$, $\forall x, y$, $(d(x) - d(y)).(x - y) \leq -\eta |x - y|^2$;
  - b is bounded Lipschitz by B ;
- polynomial growth property for d ;
- σ is Lipschitz, invertible, both σ and σ^{-1} are bounded and ${}^t\xi\sigma(x){}^t\sigma(x)\xi \geq c|\xi|^2$;
- $|\psi(x, 0)| \leq M_\psi$ and ψ Lipschitz.

Then there exist a locally Lipschitz function v, $Z \in \mathscr{M}^2(\mathbb{R}_+, \mathbb{R}^d)$ and a real λ such that, defining $Y_t^x := v(X_t^x)$, (Y^x, Z^x, λ) is a solution of the EBSDE with zero Neumann boundary conditions. Furthermore : $|v(x)| \leq C(1 + |x|^2)$. $Z_t^x = \xi(X_t^x)$ and $|\xi(x)| \leq C(1 + |x|^2)$.

Property

Uniqueness of λ.

Ergodic BSDE with Neumann boundary conditions

Problem solved :

$$Y_t^{x,0} = Y_T^{x,0} + \int_t^T [\psi(X_s^x, Z_s^{x,0}) - \lambda^0]\mathrm{d}s - \int_t^T Z_s^{x,0}\mathrm{d}W_s, \quad 0 \le t \le T + \infty$$

Problem to solve, μ is fixed

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda]\mathrm{d}s + \int_0^t [g(X_s^x) - \mu]\mathrm{d}K_s^x - \int_t^T Z_s^x\mathrm{d}W_s, \quad 0 \le t \le T + \infty$$

$$\widehat{Y}_t^x := Y_t^{x,0} - \int_0^t [g(X_s^x) - \mu]\mathrm{d}K_s^x, \quad t \ge 0$$

$\Longrightarrow (\widehat{Y}^x, Z^{x,0}, \lambda)$ is solution of the EBSDE with Neumann boundary conditions.

Ergodic BSDE with Neumann boundary conditions

Problem to solve : λ is fixed

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda]\mathrm{d}s + \int_0^t [g(X_s^x) - \mu]\mathrm{d}K_s^x - \int_t^T Z_s^x \mathrm{d}W_s, \quad 0 \le t \le T + \infty$$

$$\widehat{Y}_t^x := Y_t^{x,0} + (\lambda - \lambda^0)t - \int_0^t [g(X_s^x) - \mu]\mathrm{d}K_s^x$$

$\Longrightarrow (\widehat{Y}^x, Z^0, \mu)$ is solution of the EBSDE with Neumann boundary conditions.

Link with PDEs

Initial problem

$$\begin{cases} X_t^x = x + \int_0^t (d+b)(X_s^x)\mathrm{d}s + \int_0^t \sigma(X_s^x)\mathrm{d}W_s + \int_0^t \nabla\phi(X_s^x)\mathrm{d}K_s^x, & t \geq 0; \\ K_t^x = \int_0^t \mathbb{1}_{\{X_s^x \in \partial G\}}\mathrm{d}K_s^x, \quad K_\cdot^x \text{ is non decreasing;} \end{cases}$$

$$Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x) - \lambda]\mathrm{d}s - \int_t^T Z_s^x \mathrm{d}W_s, \qquad 0 \leq t \leq T + \infty.$$

$$Y_t^x = v(X_t^x)$$

PDE with zero Neumann boundary condition

$$\begin{cases} \mathscr{L}v(x) + \psi(x, {}^t\nabla v(x)\sigma(x)) = \lambda, & x \in G, \\ \frac{\partial v}{\partial n}(x) = 0, & x \in \partial G, \end{cases}$$

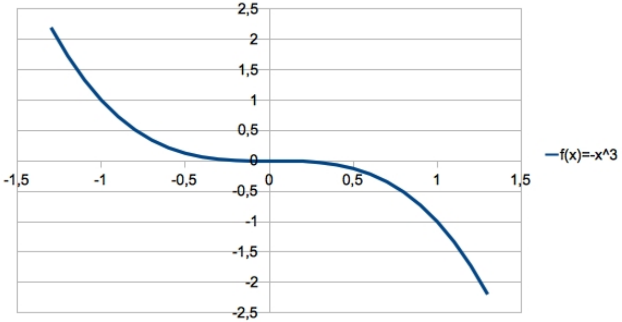
where :

$$\mathscr{L}u(x) = \frac{1}{2}\mathrm{Tr}(\sigma(x)^t\sigma(x)\nabla^2 u(x)) + {}^tf(x)\nabla u(x).$$

An example in one dimension

$$\begin{cases} X_t^x = x + \int_0^t f(X_s^x)\mathrm{d}s + \int_0^t \sigma(X_s^x)\mathrm{d}W_s + \int_0^t \nabla\phi(X_s^x)\mathrm{d}K_s^x, \quad t \geq 0; \\ K_t^x = \int_0^t \mathbb{1}_{\{X_s^x \in \partial G\}}\mathrm{d}K_s^x, \quad K_\cdot^x \text{ is non decreasing;} \end{cases}$$

If $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto -x^3 \end{cases}$



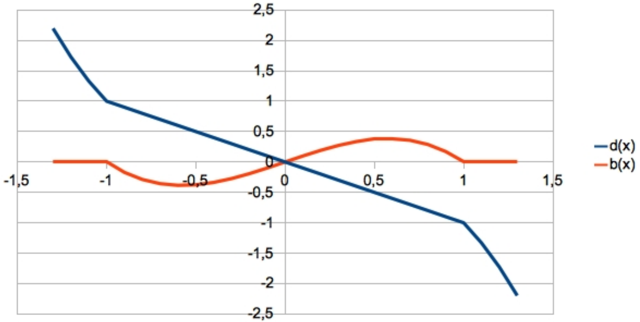
An example in one dimension

$$f(x) = d(x) + b(x).$$

where :

$$d(x) = -x^3 \mathbb{1}_{|x| \geq 1} - x \mathbb{1}_{|x| < 1}$$

$$b(x) = -x^3 \mathbb{1}_{|x| < 1} + x \mathbb{1}_{|x| < 1}$$



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