Time discretization and simulation of quadratic BSDEs

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The aim is to study the time discretization of the (decoupled) forward backward system

$$\begin{aligned} X_t &= x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, \quad 0 \leqslant t \leqslant T, \\ Y_t &= g(X_T) + \int_t^T f(Z_s)dr - \int_t^T Z_s dW_s, \quad 0 \leqslant t \leqslant T, \end{aligned}$$

when f has a quadratic growth with respect to z.
Standard assumption : f is assumed to be locally Lipschitz

$$\left|f(z) - f(z')\right| \leqslant K \left|z - z'\right| \left(1 + |z| + \left|z'\right|\right).$$

Assumptions

- g Lipschitz and bounded,
- $\sigma(t, x)$ Lipschitz and unbounded.

Other assumptions :

- When g is not smooth see [R. 2011], [E. Gobet - P. Turkedjiev Preprint],
- when g is unbounded see [R. 2012].

Existence and uniqueness

Thanks to [Kobylanski 2000] we have :

- Since g is bounded there exists a solution (Y, Z) such that Y is bounded.
- Since f is locally Lipschitz we have a uniqueness result among bounded solutions.
- $\left(\int_0^t Z_s dW_s\right)_{t\in[0,T]}$ is a BMO martingale :

$$\|Z * W\|_{BMO}^2 = \sup_{0\leqslant\tau\leqslant T \text{ stopping time}} \mathbb{E}_\tau \left[\int_\tau^T |Z_s|^2 ds\right] < +\infty.$$

The linearization trick

Let us consider two solutions (Y^1, Z^1), (Y^2, Z^2) for two terminal conditions g_1, g_2 and two generators f_1, f_2. We denote

$$\delta Y := Y^1 - Y^2, \quad \delta Z := Z^1 - Z^2, \quad \delta g := g_1 - g_2, \quad \delta f := f_1 - f_2.$$

We have

$$\delta Y_t = \delta g(X_T) + \int_t^T \delta f(Z_s^1) ds - \int_t^T \delta Z_s \left(dW_s - \gamma_s ds \right),$$

with

$$\gamma_s = \delta Z_s \frac{f_2(Z_s^1) - f_2(Z_s^2)}{|\delta Z_s|^2}.$$

Why BMO martingales are nice ?

Let us denote $\mathcal{E}(\gamma)$ the Doléans-Dade exponential associated to the martingale $(\int_0^t \gamma_s dW_s)_t$. Since $|\gamma_s| \leqslant C(1 + |Z_s^1| + |Z_s^2|)$, we have

$$\|\gamma * W\|_{BMO}^2 \leqslant C(1 + \left\|Z^1 * W\right\|_{BMO}^2 + \left\|Z^2 * W\right\|_{BMO}^2) < +\infty.$$

- $\mathcal{E}(\gamma)$ is a martingale.
- $\mathcal{E}(\gamma) \in L^p$ with $p > 1$ that depends only on $\|\gamma * W\|_{BMO}$.

Comparison and stability

- Comparison :

$$\delta Y_t = \mathbb{E}_t^{\mathbb{Q}}\left[\delta g(X_T) + \int_t^T \delta f(Z_s^1)ds\right].$$

- Stability :

$$|\delta Y_t|^q \leqslant C\mathbb{E}_t\left[|\delta g(X_T)|^q + \left|\int_t^T \delta f(Z_s^1)ds\right|^q\right].$$

Time discretization scheme

Let us consider a grid $0 = t_0 < t_1 < ... < t_n = T$ with $h_i = t_{i+1} - t_i$ and $h = \max_i h_i$. $(X_i^n)_i$ discrete approximation of X with "good" convergence properties. $(Y_i^n, Z_i^n)_i$ solution of the scheme

$$\begin{cases} Y_n^n = g(X_n^n) \\ Y_i^n = \mathbb{E}_{t_i}[Y_{i+1}^n + h_i f(Z_i^n)] \\ Z_i^n = \mathbb{E}_{t_i}[Y_{i+1}^n H_i] \end{cases}$$

with $H_i = \frac{W_{t_{i+1}} - W_{t_i}}{h_i} = \frac{\Delta W_i}{h_i}$.

Linearization of time discretization schemes

Let us consider two discretized solutions (Y^1, Z^1), (Y^2, Z^2) for two terminal conditions g_1, g_2 and two generators f_1, f_2.

$$\delta Y := Y^1 - Y^2, \quad \delta Z := Z^1 - Z^2, \quad \delta g := g_1 - g_2, \quad \delta f := f_1 - f_2.$$

We have

$$\begin{aligned} \delta Y_i &= \mathbb{E}_{t_i}[\delta Y_{i+1} + h_i(f_1(Z_i^1) - f_2(Z_i^1)) + h_i(f_2(Z_i^1) - f_2(Z_i^2))] \\ &= \mathbb{E}_{t_i}[\delta Y_{i+1} + h_i \delta f(Z_i^1) + h_i \gamma_i \delta Z_i], \end{aligned}$$

with

$$\gamma_i = \delta Z_i \frac{f_2(Z_i^1) - f_2(Z_i^2)}{|\delta Z_i|^2}.$$

Since, $\delta Z_i = \mathbb{E}_{t_i}[H_i \delta Y_{i+1}]$, we have

$$\begin{aligned} \delta Y_i &= \mathbb{E}_{t_i}[(1 + h_i \gamma_i H_i)(\delta Y_{i+1} + h_i \delta f(Z_i^1))] \\ &= \mathbb{E}_{t_i}\left[\prod_{j=i}^{n-1}(1 + h_j \gamma_j H_j)\left(\delta g(X_n^n) + \sum_{k=i}^{n-1} h_k \delta f(Z_k^1)\right)\right]. \end{aligned}$$

New assumptions for comparison and stability

- $E_t = \prod_{t_j \leqslant t}(1 + h_j \gamma_j H_j)$ is the Doléans-Dade exponential of the martingale $M_t := \sum_{t_i \leqslant t} h_i \gamma_i H_i$.
- To have $E_t \geqslant 0$, we need to have $(\gamma_i)_i$ and $(H_i)_i$ bounded.
- We take $H_i^R = \frac{\rho_R(\Delta W_i)}{h_i}$ with R well chosen.
- For γ_i we need to truncate f.

Truncation of the initial BSDE

Let us denote (Y^N, Z^N) the solution of the BSDE

$$Y_t^N = g(X_T) + \int_t^T f(\rho_N(Z_s^N))ds - \int_t^T Z_s^N dW_s,$$

and (Y^π, Z^π) the solution of the scheme

$$\begin{cases} Y_n^\pi = g(X_n^n) \\ Y_i^\pi = \mathbb{E}_{t_i}[Y_{i+1}^\pi + h_i f(\rho_N(Z_i^\pi))] \\ Z_i^\pi = \mathbb{E}_{t_i}[Y_{i+1}^\pi H_i^R]. \end{cases}$$

N and R will depend on n.

Error due to the truncation

[P. Imkeller - G. dos Reis 2010], [A. R. 2012]

For all $q > 0$, there exists $C_q > 0$ such that

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\left|Y_t - Y_t^N\right|^2\right] + \mathbb{E}\left[\int_0^T \left|Z_s - Z_s^N\right|^2 ds\right] \leqslant \frac{C_q}{N^q}.$$

Application of the comparison result

By taking R and N such that

$$E_t = \prod_{t_j \leqslant t} (1 + h_j \gamma_j^N H_j^R) > 0$$

we obtain a comparison theorem.

Corollary

$|Y^\pi| \leqslant C$ with C that does not depend on n, N, R.

Stability

We will study the error between (Y^N, Z^N) and (Y^π, Z^π) by using our stability result on schemes. We need to write the initial BSDE as a perturbed time discretization scheme.

$$\begin{cases} Y^N_{t_n} & = & g(X_T) \\ Y^N_{t_i} & = & \mathbb{E}_{t_i}[Y^N_{t_{i+1}} + \int_{t_i}^{t_{i+1}} f(\rho_N(Z^N_s))ds] \\ & = & \mathbb{E}_{t_i}[Y^N_{t_{i+1}} + h_i \left(f(\rho_N(\bar{Z}^N_{t_i})) + \zeta_i \right)] \\ \bar{Z}^N_{t_i} & = & \mathbb{E}_{t_i}[Y^N_{t_{i+1}} H^R_i] \end{cases}$$

with

$$\zeta_i := \frac{1}{h_i} \mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} \left(f(\rho_N(Z^N_s)) - f(\rho_N(\bar{Z}^N_{t_i})) \right) ds \right].$$

Stability

If we apply the linearization trick to (Y^N, Z^N) and (Y^π, Z^π) we obtain

$$Y^N_{t_i} - Y^\pi_i = \mathbb{E}_{t_i}\left[\prod_{j=i}^{n-1}(1 + h_j\gamma_j^N H_j^R)\left(g(X_T) - g(X_n^n) + \sum_{k=i}^{n-1} h_k\zeta_k\right)\right].$$

Proposition

$M_t := \sum_{t_i \leqslant t} h_i\gamma_i^N H_i^R$ is a BMO martingale. Moreover, $\|M\|_{BMO}$ is bounded by a constant that does not depend n, N and R.
Finally, there exists $q > 1$ independent of N, n and R such that

$$\left|Y^N_{t_i} - Y^\pi_i\right|^q \leqslant \mathbb{E}_{t_i}\left[|g(X_T) - g(X_n^n)|^q + \left|\sum_{k=i}^{n-1} h_k\zeta_k\right|^q\right].$$

An explicit speed of convergence

Theorem

- $h_i = T/n = h$,
- $H_i^R = \frac{\rho_R(\Delta W_i)}{h}$ with $R = \log n$,
- $N = n^{1/4}$.

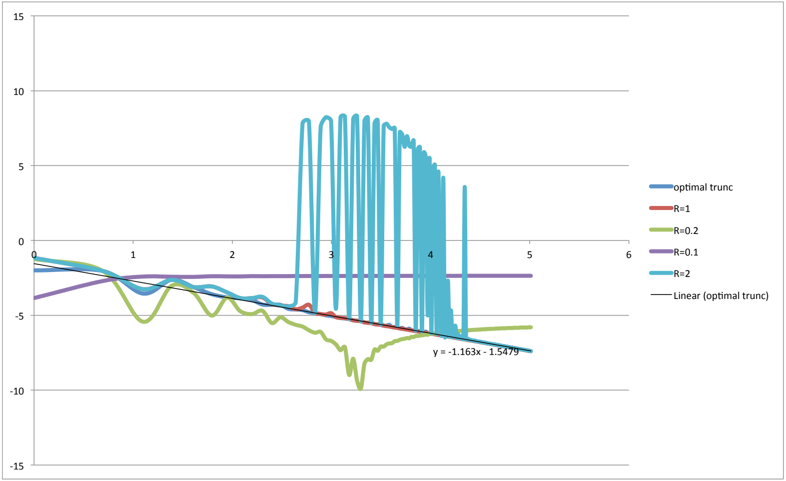
Then, for all $\eta > 0$ we have

$$\mathbb{E}\left[\sup_{0\leqslant i\leqslant n} |Y_{t_i} - Y_i^\pi|^2\right] + \mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} |Z_s - Z_i^\pi|^2\, ds\right] \leqslant C_\eta h^{1-\eta}.$$

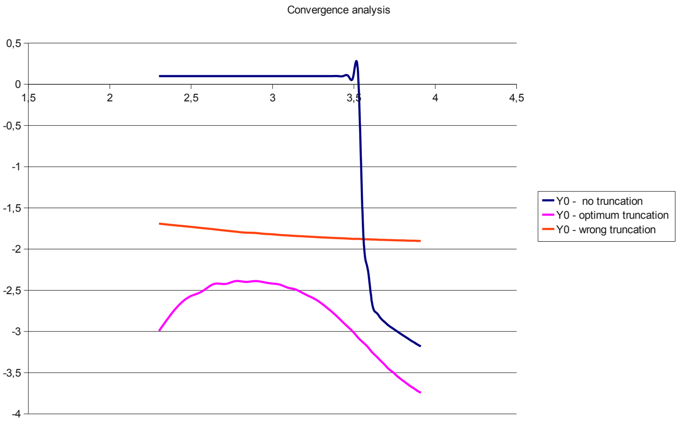
Example

- X is a geometric Brownian motion without drift (dimension 1),
- $g(x) = \sin^2(x)$,
- $f(z) = az^2$ with $a = 5$ or $a = 6$,
- n from 10 to 50,
- conditional expectation approximated by tree method or quantification method.

We know the real solution.



15
10
5
0
-5
-10
-15
0
1
2
3
4
5
6
optimal trunc
R=1
R=0.2
R=0.1
R=2
Linear (optimal trunc)
y = -1.163x - 1.5479


Convergence analysis
0,5
0
-0,5
-1
-1,5
-2
-2,5
-3
-3,5
-4
1,5
2
2,5
3
3,5
4
4,5
Y0 - no truncation
Y0 - optimum truncation
Y0 - wrong truncation