On market models that do not admit an ELMM but satisfy weaker forms of no-arbitrage

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Introductory remarks

- Absence of arbitrage is one of the fundamental notions in quantitative finance for pricing, hedging and also portfolio optimization.

- A basic step in the theoretical development (*completed mainly in work by Delbaen and Schachermayer, see also Kabanov*) was the equivalence of the economic notion of NFLVR (*no-free-lunch-with-vanishing-risk*) and the mathematical notion of ELMM *equivalent local martingale measure*) (in the general case $E\sigma MM$).

  → *For an extension to the case with short sales prohibitions see Pulido'13*

Introductory remarks

- More recently, particularly in the (descriptive) Stochastic Portfolio Theory (Fernholz, Karatzas) it was argued that the behavior in real markets corresponds to weaker notions of no-arbitrage than NFLVR.
- In parallel, the Benchmark approach to quantitative finance (Platen et al) aims at showing how pricing, hedging as well as portfolio optimization can be performed also without existence of an ELMM

Introductory remarks

Various weaker notions of no-arbitrage have therefore been introduced more recently and their consequences on pricing, hedging and portfolio optimization have been studied (see a survey in Fontana'13).

- While NFLVR is not robust with respect to changes in numeraire and reference filtration, the weaker concepts are.
- We want to concentrate here on the weaker concepts of NUPBR (no-unbounded-profit-with-bounded-risk) (Karatzas-Kardaras'07) and the equivalent one of NA1 (no-arbitrage-of-the-first-kind) (Kardaras'12) that appear as minimal conditions to meaningfully solve portfolio optimization problems.

Introductory remarks

- Working under NA1 (NUPBR) one cannot anymore rely on an ELMM nor on the corresponding density process (R.-N.-derivative)

  $\longrightarrow$ *Beyond NFLVR all possible candidates for the density process of an ELMM turn out to be strict local martingales.*

Introductory remarks

A crucial concept, generalizing the density process, is that of an ELMD (equivalent local martingale deflator) or, more generally, ESMD (equivalent supermartingale deflator): over a finite horizon $[0, T]$ the latter is a process $D_t \geq 0$ with $D_0 = 1$, $D_T > 0$, $\mathbb{P} - a.s.$ and such that $D_t \bar{V}_t$ is a supermartingale for all discounted admissible (non-negative) portfolio processes $\bar{V}_t$.

- *If $\exists$ an ESMM (equivalent supermartingale measure), namely $\mathbb{Q} \sim \mathbb{P}$ for which all $\bar{V}_t$ are supermartingales, then $D_t := \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_{|\mathcal{F}_t}$ is an ESMD, actually a martingale, but an ESMD is not necessarily a density process.*
- Like the density process, an ESDM is itself a supermartingale, but it may fail to be a martingale, even a local martingale.

Introductory remarks

- The interest therefore arises in finding market models that fall between NFLVR and NA1: they allow for classical arbitrage, but make it still possible to perform pricing, hedging and portfolio optimization.
- *For continuous market models a classical example is related to Bessel processes: it appears already in Delbaen-Schachermayer'95 and was further developed by various authors (e.g. Platen, Ruf, Hulley,..)*

There is therefore interest in finding other models, beyond Bessel processes, that satisfy NA1 but not NFLVR and whether there exists a systematic procedure to generate such models in a more general semimartingale framework; equivalently, as we shall see, to generate ESMDs that are strict supermartingales.

$\rightarrow$ *This is the aim of the first part below*

Introductory remarks

As mentioned in Christensen-Larsen'07 and Hulley-Platen'10, there may not be many possibilities while remaining within continuous market models (*one basically remains within time changed Bessel processes*); more possibilities may arise in discontinuous market models and/or in models with portfolio constraints (beyond standard admissibility).

- It was shown in Kardaras'09 that for exponential Levy models the various notions of no-arbitrage, weaker than NFLVR, are all equivalent. This leaves however open the case of jump-diffusion models.

Introductory remarks

- On the other hand, portfolio constraints may lead to particular situations that are worth exploring. *(Karatzas-Kardaras'07 consider the case of predictable closed convex constraints)*.

We shall be interested in exploring the specific case of jump-diffusion market models, possibly with portfolio constraints.

OUTLINE OF THE REMAINING PART

A. (Based on Ruf-R.'13)

- A systematic procedure to generate models that satisfy NA1, but not NFLVR
- An example relating to discontinuous market models

OUTLINE OF THE REMAINING PART

B. (Based on Mancin-R.'13)

- The jump-diffusion market model
- The GOP (growth optimal portfolio) as a basic tool to obtain an ESMD (*given by the inverse of the discounted GOP*) and equivalence between: validity of NA1 and existence of an ESMD
- The ESMD given by the discounted inverse of the GOP as the only candidate for the density process of an ESMM
- The case of portfolio constraints where the inverse of the discounted GOP is a strict supermartingale, not even a local martingale.

Market model

- Given a finite time horizon $T < \infty$, consider a market $((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}), S)$ with $(\mathcal{F}_t)$ right continuous and $S = (S_t) = (S_t^1, \cdots, S_t^d)$ the already discounted prices of d risky assets supposed to be general non-negative semimartingales.
- Given a self-financing, predictable strategy $H = (H_t)$, let

$$V^{x,H} = (V_t^{x,H}) = x + (H \cdot S)_t = x + \int_0^t H_u dS_u$$

be the value process corresponding to H with $V_0^{x,H} = x$.

Definition *(admissible strategy)* An $S-$integrable, predictable H is α–*admissible* if $H_0 = 0$ and $V_t^{0,H} \geq -\alpha$, $t \in [0, T]$ *a.s.* H is *admissible* if it is admissible for some $\alpha > 0$.

Market model

Definition *(arbitrage strategy)* An admissible H is an arbitrage strategy if $\mathbb{P}(V_T^{0,H} \geq 0) = 1$ and $\mathbb{P}(V_T^{0,H} > 0) > 0$. It is a strong arbitrage if $\mathbb{P}(V_T^{0,H} > 0) = 1$.

Definition *(NA1)* An $\mathcal{F}_T-$measurable random variable ξ is called an Arbitrage of the First Kind if $\mathbb{P}(\xi \geq 0) = 1$, $\mathbb{P}(\xi > 0) > 0$, and for all $x > 0$ there exists an $x-$admissible strategy H such that $V_T^{x,H} \geq \xi$. We shall say that the market admits No Arbitrage of the First Kind (NA1), if there is no arbitrage of the first kind in the market.

Market model

Definition(*NUPBR*) There is No Unbounded Profit With Bounded Risk (NUPBR) if the set

$$\mathcal{K}_1 = \left\{ V_T^{0,H} \mid H = (H_t) \text{ is a 1–admissible strategy for } S \right\}$$

is bounded in L^0, that is, if

$$\lim_{c \uparrow \infty} \sup_{W \in \mathcal{K}_1} \mathbb{P}(W > c) = 0$$

- (NA1) and (NUPBR) can be shown to be equivalent (Kardaras'10)
- (NFLVR) implies (NUPBR) but not viceversa.

Proposition 1. (Kardaras'12, Takaoka'13, see also Song'13) A market satisfies NUPBR (NA1) if and only if there exists an ESMD.

Construction of strict local martingales

- Based on Delbaen-Schachermayer'95 (*Föllmer exit measure*) we start from a space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ where, under $\mathbb{Q}$, S_t^i are local martingales (recall S_t^i are discounted)

  $\rightarrow$ *The market* $((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q}), S)$ *satisfies NFLVR.*

- Consider then a non-negative $\mathbb{Q}-$martingale $Y = (Y_t)$ with $Y_0 = 1$ and stopped at 0. Let $\tau := \inf\{t \geq 0 \mid Y_t = 0\}$ and make the

Assumption 1.

$$\mathbb{Q}(Y(T) = 0) = \mathbb{Q}(\tau \leq T) > 0 \text{ and } \mathbb{Q}(\{Y(\tau-) > 0\} \cap \{\tau \leq T\}) = 0.$$

$\rightarrow$ Assumption 1 and the martingality of Y imply $\mathbb{Q}\{\tau \leq T\} < 1$.

Construction of strict local martingales

- Being Y_t a $\mathbb{Q}-$martingale, one may generate a probability $\mathbb{P}$ via $d\mathbb{P}/d\mathbb{Q} = Y_T$
  - $\rightarrow$ $\mathbb{P}$ is absolutely continuous w.r.to $\mathbb{Q}$ but not $\mathbb{P} \sim \mathbb{Q}$.
  - $\rightarrow$ $\mathbb{P}$ *will correspond to the probability in the original market*

Lemma. (See e.g. Carr et al.'12) Under Assumption 1 the process $1/Y$ is a nonnegative $\mathbb{P}-$strict local martingale with $\mathbb{P}(1/Y(T) > 0) = 1$. Furthermore, $\mathbb{P}\{\tau < T\} = 0$.

Construction of strict local martingales

To proceed, introduce

Assumption 2. There exists $x \in (0,1)$ and an admissible strategy $H = (H_t)$ such that $V_T^{x,H} \geq \mathbf{1}_{\{Y_T > 0\}}$.

$\rightarrow$ *The assumption is equivalent to stating that the minimal superreplication price of* $\mathbf{1}_{\{Y_T > 0\}}$ *is less than* 1.

Theorem. Under Assumptions 1 and 2 the market $((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}), S)$ satisfies NA1 but not NFLVR. The H in Assumption 2 is a strong arbitrage in this market.

Construction of strict local martingales

(Sketch of the proof:)

- H from Assumption 2 is $\mathbb{Q}-$admissible and thus also $\mathbb{P}-$admissible. Since $\mathbb{P}\{\mathbf{1}_{\{Y_T>0\}} = 1\} = \mathbb{P}\{\tau \geq T\} = 1$ and $x < 1$, the strategy H is a strong arbitrage thus excluding NFLVR.
- $1/Y$ is a $\mathbb{P}-$local martingale and also S^i/Y are. There exists thus an ESMD and by Proposition 1 we have that NA1 (NUPBR) holds.

Construction of strict local martingales

This result leads to a systematic procedure since *(see Delbaen-Schachermayer'95, Ruf'13, Imkeller-Perkowski'13)* basically any market $((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}), S)$ that satisfies NA1 but not NFLVR implies the existence of a measure $\mathbb{Q}$ and of a $\mathbb{Q}-$local martingale Y that satisfies Assumption 1 and for which $d\mathbb{P}/d\mathbb{Q} = Y_T$.

Example

1. *(Adapted from Chau Ngoc Huy).* Start from a $\mathbb{Q}-$Poisson process N_t with intensity $\lambda > 1/T$. Put $Y_t := N_t - \lambda t + 1$, stopped when it first hits zero *(stopping time τ)* or when it first jumps *(random time ρ)*. Let $S^1 = Y$ and S^i for $i = 2, \cdots, d$ be arbitrary $\mathbb{Q}-$local martingales.

- It can be seen that $\mathbb{Q}(Y_T = 0) = \exp(-1)$ *(argument based on using the random times τ and ρ)* and so Assumption 1 holds.
- Furthermore, also Assumption 2 holds with $x = 1 - \exp(-1)$ and $H = (H_t^1, \cdots, H_t^d)$ where $H_t^1 = \exp(\lambda t - 1)\mathbf{1}_{\{t \leq \rho \wedge \tau\}}$ and $H_t^i = 0,\ i = 2, \cdots, d$.
  - $\rightarrow$ Therefore the above Theorem holds implying that NA1 holds but to NFLVR.

Example

2. The previous example can be generalized by considering a marked point process N_t with jump intensity $\lambda > 1/T$ and an arbitrary distribution F over the mark space $[F_{\min}, F_{\max}]$ where $F_{\min} \leq 1 \leq F_{\max}$ and F has expectation 1.

$\rightarrow$ Assumption 1 holds as before and a Assumption 2 holds with $x = \frac{F_{\max}}{F_{\min}}\left(1 - \exp\left(-\frac{1}{F_{\min}}\right)\right) < 1$ and $H_t^1 = \frac{\exp\left(-\frac{1-\lambda t}{F_{\max}}\right)}{F_{\min}} \mathbf{1}_{\{t \leq \rho \wedge \tau\}}$ and thus the above Theorem holds here as well.

Jump-diffusion market model

- On $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ let there be given d sources of randomness
- $W = \{W_t = (W_t^1, \cdots, W_t^m)'\}$ an $m-$dimensional standard Wiener $(m \leq d)$
- $N = \{N_t = (N_t^1, \cdots, N_t^{d-m})'\}$ a $(d-m)$-dimensional Poisson counting process with $\mathcal{F}_t-$intensity $\lambda = \{(\lambda_t^1, \cdots, \lambda_t^{(d-m)})'\}$
- $dM_t^k := \frac{dN_t^k - \lambda_t^k dt}{\sqrt{\lambda_t^k}}$ the associated compensated martingale

Jump-diffusion market model

- There are $d+1$ securities:

$$\begin{cases} dS_t^0 = S_t^0 r_t dt, \quad S_0^0 = 1 \\ dS_t^j = S_{t-}^j \left(a_t^j dt + \sum_{k=1}^m b_t^{j,k} dW_t^k + \sum_{k=m+1}^d b_t^{j,k} dM_t^{k-m} \right), \; S_0^j > 0 \end{cases}$$

Assumption 1:

$$b_t^{j,k} \geq -\sqrt{\lambda_t^{k-m}}, \quad \forall t \in [0,\infty), j \leq d, \; k \in \{m+1, \cdots, d\}$$

$b_t = \{b_t^{j,k}\}$ is invertible for a.e. $t \in [0, T]$

Jump-diffusion market model

- The generalized market price of risk is

$$\theta_t = (\theta_t^1, \cdots, \theta_t^d)' = b_t^{-1}[a_t - r_t \underline{1}] \quad \text{implying}$$

$$dS_t^j = S_{t-}^j \left(r_t dt + \sum_{k=1}^{m} b_t^{j,k}(\theta_t^k dt + dW_t^k) + \sum_{k=m+1}^{d} b_t^{j,k}(\theta_t^k dt + dM_t^{k-m}) \right)$$

Admissible strategies

- Let $\delta = \{\delta_t = (\delta_t^0, \delta_t^1, \cdots, \delta_t^d)'\}$ be predictable with $\int_0^T \delta_t^j dS_t^j < \infty$ and define (portfolio value corresponding to δ)

$$S_t^{s,\delta} = \sum_{j=0}^{d} \delta_t^j S_t^j \quad \text{with} \quad S_0^{s,\delta} = s > 0$$

- δ is admissible if $S_t^{s,\delta} \geq 0 \ \forall t \in [0,\infty)$ and $dS_t^{s,\delta} = \sum_{j=0}^{d} \delta_t^j dS_t^j$ (*self-financing*)
- The discounted portfolio process is $\bar{S}_t^{s,\delta} := \frac{S_t^{s,\delta}}{S_t^0}$

Admissible strategies

- The strategy expressed in terms of fractions of invested wealth is $\pi^j_{\delta,t} = \delta^j_t \frac{S^j_{t-}}{S^{s,\delta}_{t-}}$ implying

$$dS^{s,\delta}_t = S^{s,\delta}_{t-} \quad \Big\{ r_t dt + \sum_{k=1}^m \left(\sum_{j=1}^d \pi^j_{\delta,t} b^{j,k}_t \right) \left(\theta^k_t dt + dW^k_t \right) + \sum_{k=m+1}^d \left(\sum_{j=1}^d \pi^j_{\delta,t-} b^{j,k}_t \right) \left(\theta^k_t dt + dM^{k-m}_t \right) \Big\}$$

- Defining $S^\delta_t := S^{1,\delta}_t$, one has $S^{s,\delta}_t = s S^{1,\delta}_t$

Growth optimal portfolio

Definition: For an admissible δ, the growth rate $g^\delta = (g^\delta_t)$ is the drift in the SDE of $\log S^\delta = (\log S^\delta_t)$. A strategy δ^* (and the corresponding S^{δ^*}) is said to be growth optimal if $g^{\delta^*} \geq g^\delta$ for all admissible δ.

- For a generic admissible δ one has

$$g^\delta_t = r_t + \sum_{k=1}^m \left[\sum_{j=1}^d \pi^j_{\delta,t} b^{j,k}_t \theta^k_t - \frac{1}{2} \left(\sum_{j=1}^d \pi^j_{\delta,t} b^{j,k}_t \right)^2 \right]$$
$$+ \sum_{k=m+1}^d \left[\sum_{j=1}^d \pi^j_{\delta,t} b^{j,k}_t \left(\theta^k_t - \sqrt{\lambda^{k-m}_t} \right) + \log \left(1 + \sum_{j=1}^d \pi^j_{\delta,t} \frac{b^{j,k}_t}{\sqrt{\lambda^{k-m}_t}} \right) \lambda^{k-m}_t \right]$$

$\rightarrow$ *Assumption 1 guarantees that*

$$\left(1 + \sum_{j=1}^d \pi^j_{\delta,t} \frac{b^{j,k}_t}{\sqrt{\lambda^{k-m}_t}} \right) > 0$$

Growth optimal portfolio

- To maximize g^{δ}, maximize individually the two sums thereby putting

$$c_t^k := \sum_{j=1}^{d} \pi_{\delta,t}^{j} b_t^{j,k}$$

and making

Assumption 2: $\sqrt{\lambda_t^{k-m}} > \theta_t^k, \quad \forall t \in [0,\infty), k \in \{m+1, \cdots, d\}$

Growth optimal portfolio

- The maximizing values c_t^{*k} are

$$c_t^{*k} = \begin{cases} \theta_t^k & \text{for} \quad k \in \{1, 2, \cdots, m\} \\ \frac{\theta_t^k}{1 - \theta_t^k (\lambda_t^{k-m})^{-\frac{1}{2}}} & \text{for} \quad k \in \{m+1, \cdots, d\} \end{cases}$$

It follows that $\pi_{\delta_*, t} = (\pi^1_{\delta_*, t}, \cdots, \pi^d_{\delta_*, t}) = (c_t^*)' b_t^{-1}$ and

$$dS_t^{\delta_*} = S_{t-}^{\delta_*} \Bigg(r_t dt + \textstyle\sum_{k=1}^m \theta_t^k (\theta_t^k dt + dW_t^k) + \textstyle\sum_{k=m+1}^d \frac{\theta_t^k}{1 - \theta_t^k (\lambda_t^{k-m})^{-\frac{1}{2}}} (\theta_t^k dt + dM_t^{k-m}) \Bigg)$$

Basic results (analogous to continuous models)

Proposition 2: Under the given assumptions and without restrictions on the portfolio one has that

$$\hat{Z}_t := \frac{1}{\bar{S}_t^{\delta_*}}$$

is a supermartingale deflator.

In fact, by application of Ito's formula one has that

$$d\left(\frac{\bar{S}_t^{\delta}}{\bar{S}_t^{\delta_*}}\right) = \sum_{k=1}^{m}\left(\sum_{j=1}^{d}\delta_t^j\hat{S}_t^j b_t^{j,k} - \hat{S}_t^{\delta}\theta_t^k\right)dW_t^k$$
$$+\sum_{k=m+1}^{d}\left(\left(\sum_{j=1}^{d}\delta_t^j\hat{S}_{t-}^j b_t^{j,k}\right)\left(1-\frac{\theta_t^k}{\sqrt{\lambda_t^{k-m}}}\right)-\hat{S}_{t-}^{\delta}\theta_t^k\right)dM_t^{k-m}$$

Basic results

Proposition 3: There is equivalence between

i) Existence of an ESMD

ii) Validity of NA1 (NUPBR)

- This statement is shown in Karatzas-Kardaras'07 also in presence of predictable closed convex constraints *(see also Takaoka'13, Song'13 in a general setting but without portfolio constraints)*

Basic results

Proposition 4: Under Assumptions 1 and 2, and without portfolio restrictions, the process $\hat{Z}_t$ is the only candidate for the density process of an ESMM.

- Particularizing the expression for $d\left(\frac{\bar{S}_t^{\delta}}{\bar{S}_t^{\delta *}}\right)$ to the case of $\delta = (1, 0, \cdots, 0)$ one obtains

$$d\left(\frac{1}{\bar{S}_t^{\delta *}}\right) = -\frac{1}{\bar{S}_t^{\delta *}} \sum_{k=1}^{m} \theta_t^k dW_t^k - \frac{1}{\bar{S}_{t-}^{\delta *}} \sum_{k=m+1}^{d} \theta_t^k dM_t^{k-m}$$

- We shall next show that, under Assumption 2,

$$dL_t = -L_{t-}\left(\sum_{k=1}^{m} \theta_t^k dW_t^k + \sum_{k=m+1}^{d} \theta_t^k dM_t^{k-m}\right)$$

Density process

- The general formula for the R.-N.-derivative $L_t := \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_{|\mathcal{F}_t}$ of an absolutely continuous measure transformation in the jump-diffusion case is

$$L_t = \exp \left\{-\frac{1}{2}\sum_{k=1}^{m}\int_0^t \left(\theta_s^k\right)^2 ds - \sum_{k=1}^{m}\int_0^t \theta_s^k dW_s^k\right\} \prod_{k=m+1}^{d}\left\{\exp\left[\int_0^t \theta_t^k \sqrt{\lambda_s^{k-m}} ds\right] \prod_{n=1}^{N_t^{k-m}}\left(1 - \frac{\theta_{T_n}^k}{\sqrt{\lambda_{T_n}^{k-m}}}\right)\right\}$$

$\rightarrow$ *Assumption 2 guarantees that* $\left(1 - \frac{\theta_{T_n}^k}{\sqrt{\lambda_{T_n}^{k-m}}}\right) > 0$.

Therefore, if Assumption 2 does not hold, there cannot exist an ESMM.

Density process

- Imposing that Q be an ESMM one obtains

$$\begin{cases} \varphi_t^k &= -\theta_t^k \qquad \text{for} \quad k \in \{1, 2, \ldots, m\} \\ \psi_t^{k-m} &= 1 - \frac{\theta_t^k}{\sqrt{\lambda_t^{k-m}}} \qquad \text{for} \quad k \in \{m+1, \ldots, d\}. \end{cases}$$

and with it the required dynamics for L_t.

Summing up (unconstrained case)

Under Assumptions 1 and 2 we have obtained the following:

- The density process of an ESMM is an ESMD;
- $\hat{Z}_t$ exists and is the only ESMD;
- $\hat{Z}_t$ is the only candidate for the density process of an ESMM.

  → *If $\hat{Z}_t$ is a strict supermartingale, then there does not exist an ESMM and thus also no NFLVR. However, since $\hat{Z}_t$ is an ESMD, the properties NA1 (NUPBR) still hold.*

- Furthermore, whenever Assumption 2 does not hold then, independently of the presence of portfolio restrictions, there does not exists an ESMM and so we do not have NFLVR.

Constraints on the portfolio

- For simplicity we consider the case of $d = 2$ ($\pi_t = (\pi_t^0, \pi_t^1, \pi_t^2)'$) and discuss two possible situations A. and B.

Case A. Assumption 2 is not required to be verified, but we require the condition

$$(R) \quad \pi_t^1 b_t^{1,2} + \pi_t^2 b_t^{2,2} \leq C \quad \text{for some real} \quad C > 0$$

Constraints on the portfolio

- Under absence of Assumption 2 the existence of the GOP is not guaranteed without restrictions on the portfolio strategy. *In fact, the growth rate would go to infinity for* $\pi_t^i b_t^{1,2} \to \infty$.
  - $\to$ On the other hand, restriction (R) guarantees the existence of the GOP.
- *With restriction (R) we have in fact*

$$\tilde{c}_t^k = \begin{cases} \theta_t^1 & \text{for} \quad k = 1 \\ C & \text{for} \quad k = 2. \end{cases}$$

and

$$d\bar{S}_t^{\delta_*} = \bar{S}_{t-}^{\delta_*} \left\{ \theta_t^1 \left(\theta_t^1 dt + dW_t \right) + C \left(\theta_t^2 dt + dM_t \right) \right\}$$

Constraints on the portfolio

Proposition 5: Under Assumption 1 and restriction (R) the process $\hat{Z}_t := (\bar{S}_t^{\delta_*})^{-1}$ as well as the processes $\bar{S}_t^{\delta}(\bar{S}_t^{\delta_*})^{-1}$, for admissible δ, are supermartingales that are not local martingales.

- Show only the case of $\hat{Z}_t$. We have

$$d\hat{Z}_t = -\hat{Z}_t \left(C \left(\theta_t^2 - \sqrt{\lambda_t} \right) + \frac{C\lambda_t}{\sqrt{\lambda_t}+C} \right) dt - \frac{1}{\bar{S}_{t-}^{\delta_*}} \left(\theta_t^1 dW_t + \frac{C\sqrt{\lambda_t}}{\sqrt{\lambda_t}+C} dM_t \right)$$

that has a strictly negative drift if, violating Assumption 2, we have $\theta_t^2 > \sqrt{\lambda_t}$.

$\rightarrow$ $\hat{Z}_t$ is a supermartingale deflator and so NA1 (NUPBR) holds. On the other hand, without Assumption 2, we have already seen that NFLVR does not hold.

Constraints on the portfolio

Case B. Assumptions 1 and 2 both hold as well as restriction (R).

Proposition 6: Under Assumptions 1 and 2 and restriction (R), if $C < \frac{\theta_t^2}{1-\frac{\theta_t^2}{\sqrt{\lambda_t}}}$, then the same conclusions as in Proposition 5 hold.

- Again we show only the case of $\hat{Z}_t$: here the drift in $d\hat{Z}_t$ is

$$-\hat{Z}_t\left(C\left(\theta_t^2-\sqrt{\lambda_t}\right)+\frac{C\lambda_t}{\sqrt{\lambda_t}+C}\right)$$
$$=\hat{Z}_t\left(\frac{C\left(C\left(\theta_t^2-\sqrt{\lambda_t}\right)+\theta_t^2\sqrt{\lambda_t}\right)}{\sqrt{\lambda_t}+C}\right)$$

and it is strictly negative since $C>0$ and $C < \frac{\theta_t^2}{1-\frac{\theta_t^2}{\sqrt{\lambda_t}}}$.

Constraints on the portfolio

- Also in the present case B., $\hat{Z}_t$ is a supermartingale deflator and so NA1 (NUPBR) hold.
- *With Assumption 2 in force, $\hat{Z}_t$ is the only candidate to be the density process of an ELMM. However, being $\hat{Z}_t$ a strict supermartingale, it cannot be a density process and so NFLVR does not hold.*

Thank you for your attention