Investment strategies under debt borrowing limit constraints

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Related papers and our motivation

Related papers of optimal investment timing problem for:

1. the firm financed by all-equity without financing constraint: McDonald and Siegel (1986, QJE):
2. the firm financed by all-equity with financing constraint: Boyle and Guthrie (2003, JF):
   - Investment thresholds are non-monotonic with the friction
3. the firm financed by bank debt without financing constraint: Sundaresan and Wang (2007, AER):
   - Investment thresholds for levered firm are smaller than those for unlevered firm
4. the firm financed by market debt with financing constraint: Shibata and Nishihara (2012, JBF):
   - Investment thresholds have a U-shaped curve with the friction

In this talk, we consider the optimal investment timing problem for

5. the firm financed by bank and market debt with financing constraints

We examine how financing constraints influence investment timing, quantity, debt structure (bank debt or market debt), default probability, and credit spreads.

Contents

1. Model setup and value functions
2. (P1) Investment problem under all-equity financing: McDonald and Siegel (1986, QJE)
3. (P2) Investment problem under debt financing (without constraints): Sundaresan and Wang (2007, AER)
4. (P3) Investment problem under debt financing with constraints: our model
5. Model implication
6. Concluding remarks

Model setup

- A firm possesses an investment opportunity
- δX_t: cash inflow after investment
  - δ: quantity
  - X_t: price

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}z_t, \; X_0 = x > 0, \tag{1}$$

where $\mu \in (0, r)$, $\sigma > 0$, and $(z_t)_{t \geq 0}$: standard Brownian motion.

- $I(\delta) > 0$: investment cost expenditure with

$$I(0) > 0, \quad I'(\delta) > 0, \quad I''(\delta) > 0 \tag{2}$$

- The firm possesses three type of financing structures:
  - all-equity financing
  - market debt financing
  - bank debt financing
- $r > 0$: risk-neutral discount factor.

Control variables

Firm's control variables:

- Before investment, the firm decides
  - investment threshold (i.e., investment timing)
  - investment quantity
  - coupon payment under debt financing
  - debt structure (bank debt or market debt)
- After investment, the firm determines
  - bankruptcy threshold (bankruptcy timing).

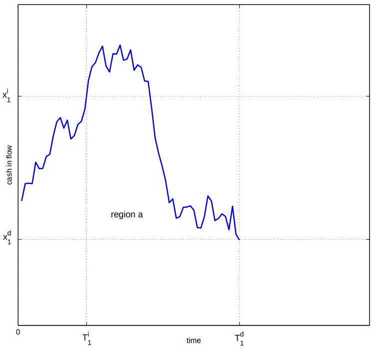
Difference between bank and market debt

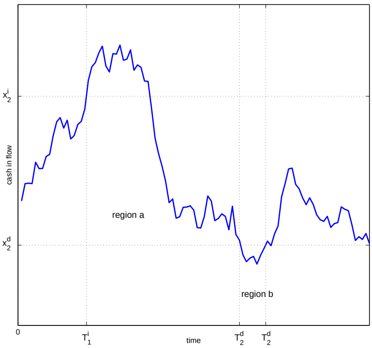
The only difference between bank and market debt is bankruptcy procedure:

- Under market debt financing, coupon payments to the market lender cannot changed outside of the formal bankruptcy process. See e.g., Leland, (1994, JF), Leland and Toft (1996, JF), and many papers...
- Under bank debt financing, coupon payments to the bank lender are reduced in the course of a costless private workout. See, e.g., Mella-Barrel Perraudin (1997, JF), Fan and Sundaresan (2000, RFS), and Hackbarth, et al. (2007, RFS).

These assumptions are the same as in Gertner and Scharfstein (1991, JF), Hart and Moore (1995,AER), Bolton and Freixas (2000, JPE), Cantille and Wright (2000, RFS), and Hackbarth et al. (2007, RFS).

Market debt and bank debt





"$j = 1$" market debt financing

"$j = 2$" bank debt financing

- $T_j^{\mathrm{i}} := \inf\{t \geq 0; X_t \geq x_j^{\mathrm{i}}\}$ where x_j^{i}: investment threshold.
- $T_j^{\mathrm{d}} = \inf\{t \geq T_j^{\mathrm{i}}; X_t \leq x_j^{\mathrm{d}}\}$ where x_j^{d}: default threshold.

Equity value under market debt financing: $E_1^{\mathrm{a}}(X_t, c_1, \delta_1)$

For any $t > T_1^{\mathrm{i}}$,

$$E_1^{\mathrm{a}}(X_t, c_1, \delta_1) = \sup_{T_1^{\mathrm{d}} \geq t > T_1^{\mathrm{i}}} \mathbb{E}_t \left[\int_t^{T_1^{\mathrm{d}}} \mathrm{e}^{-r(u-t)} (1-\tau)(\delta_1 X_u - c_1) \mathrm{d}u \right], \qquad (3)$$

where

- subscript "1" indicates the market debt financing.
- superscript "a" represents the value after investment.
- superscripts "i" and "d" indicates the investment and default strategies, respectively.
- $\tau > 0$: tax rate.
- $c_1 \geq 0$: coupon payment for market debt.

$E_1^{\rm a}(X_t, c_1, \delta_1)$ is rewritten as

$$E_1^{\rm a}(X_t, c_1, \delta_1) \tag{4}$$
$$= \max_{x_1^{\rm d} \geq 0} \Pi \delta_1 X_t - (1-\tau)\frac{c_1}{r} - \left\{ \Pi \delta_1 x_1^{\rm d} - (1-\tau)\frac{c_1}{r} \right\} \left(\frac{X_t}{x_1^{\rm d}} \right)^{\gamma},$$

where

$$\Pi := \frac{1-\tau}{r-\mu} > 0, \tag{5}$$
$$\gamma := \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} < 0, \tag{6}$$
$$\beta := \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1. \tag{7}$$

Optimal bankruptcy threshold (maximizing (4) with x_1^{d} gives):

$$x_1^{\mathrm{d}}(c_1, \delta_1) = \underset{x_1^{\mathrm{d}}}{\operatorname{argmax}}\, E_1^{\mathrm{a}}(X_t, c_1, \delta_1) = \kappa_1^{-1} \frac{c_1}{\delta_1}, \tag{8}$$

where

$$\kappa_1 = \frac{\gamma - 1}{\gamma} \frac{1}{1 - \tau} \Pi r > 0. \tag{9}$$

Note that

- $x_1^{\mathrm{d}}(c_1, \delta_1)$ is a linear function of c_1.
- $\lim_{c_1 \downarrow 0} x_1^{\mathrm{d}}(c_1, \delta_1) = 0$.
- $\lim_{c_1 \downarrow 0} E_1^{\mathrm{a}}(X_t, c_1, \delta_1) = \Pi \delta_1 X_t$ (due to $\gamma < 0$).
- These results are given by Black and Cox (1976, JF).

Market debt value: $D_1^{\mathrm{a}}(X_t, c_1, \delta_1)$

For any $t > T_1^{\mathrm{i}}$,

$$D_1^{\mathrm{a}}(X_t, c_1, \delta_1) = \mathbb{E}_t\left[\int_t^{T_1^{\mathrm{d}}} \mathrm{e}^{-r(u-t)} c_1 \mathrm{d}u + \mathrm{e}^{-r(T_1^{\mathrm{d}}-t)}(1-\alpha)\Pi\delta_1 x_1^{\mathrm{d}}(c_1, \delta_1)\right] \tag{10}$$

where $\alpha \in (0,1)$: bankruptcy cost.
Here, $D_1^{\mathrm{a}}(X_t, c_1, \delta_1)$ is written as

$$D_1^{\mathrm{a}}(X_t, c_1, \delta_1) = \frac{c_1}{r} - \left\{\frac{c_1}{r} - (1-\alpha)\Pi\delta_1 x_1^{\mathrm{d}}(c_1, \delta_1)\right\}\left(\frac{X_t}{x_1^{\mathrm{d}}(c_1, \delta_1)}\right)^{\gamma}, \tag{11}$$

Note that

- $\lim_{c_1 \downarrow 0} D_1^{\mathrm{a}}(X_t, c_1, \delta_1) = 0$ (due to $\lim_{c_1 \downarrow 0} x_1^{\mathrm{d}}(c_1, \delta_1) = 0$ and $\gamma < 0$).

Total firm value under market debt: $V_1^{\mathrm{a}}(X_t, c_1, \delta_1)$

For any $t > T_1^{\mathrm{i}}$,

$$\begin{aligned}\underbrace{V_1^{\mathrm{a}}(X_t, c_1, \delta_1)}_{\text{Levered total firm value}} &= E_1^{\mathrm{a}}(X_t, c_1, \delta_1) + D_1^{\mathrm{a}}(X_t, c_1, \delta_1) \\ &= \underbrace{\Pi\delta_1 X_t}_{\text{Unlevered total firm value}} + \underbrace{\tau\frac{c_1}{r}\left(1 - \left(\frac{X_t}{x_1^d(c_1, \delta_1)}\right)^{\gamma}\right)}_{\text{Tax benefit}} \\ &\quad \underbrace{-\alpha\Pi\delta_1 x_1^{\mathrm{d}}(c_1, \delta_1)\left(\frac{X_t}{x_1^{\mathrm{d}}(c_1, \delta_1)}\right)^{\gamma}}_{\text{Bankruptcy cost}}, \qquad (12)\end{aligned}$$

Note that

- Total firm value in (12) has three components.
- $\lim_{c_1 \downarrow 0} V_1^{\mathrm{a}}(X_t, c_1, \delta_1) = \lim_{c_1 \downarrow 0} E_1^{\mathrm{a}}(X_t, c_1, \delta_1) = \Pi\delta_1 X_t$ (due to $\lim_{c_1 \downarrow 0} x_1^{\mathrm{d}}(c_1, \delta_1) = 0$ and $\gamma < 0$).

Equity value under bank debt financing: $E_2^{\mathrm{a}}(X_t, c_2, \delta_2)$

$E_2^{\mathrm{a}}(X_t, c_2, \delta_2)$ and $E_2^{\mathrm{b}}(X_t, c_2, \delta_2)$:

$$E_2^{\mathrm{a}}(X_t, c_2, \delta_2) = \sup_{T_2^{\mathrm{d}} \geq t > T_2^{\mathrm{i}}} \mathbb{E}_t \Bigg[\int_t^{T_2^{\mathrm{d}}} \mathrm{e}^{-r(u-t)}(1-\tau)(\delta_2 X_u - c_2)\mathrm{d}u + \mathrm{e}^{-r(T_2^{\mathrm{d}}-t)} E_2^{\mathrm{b}}(X_{T_2^{\mathrm{d}}}, c_2, \delta_2) \Bigg], \quad (13)$$

where

$$E_2^{\mathrm{b}}(X_t, c_2, \delta_2) = \mathbb{E}_t \Bigg[\int_t^{T_2^{\mathrm{d}}} \mathrm{e}^{-r(u-t)}(1-\tau)(\delta_2 X_u - s(X_u, \delta_2))\mathrm{d}u + \mathrm{e}^{-r(T_2^{\mathrm{d}}-t)} E_2^{\mathrm{a}}(X_{T_2^{\mathrm{d}}}, c_2, \delta_2) \Bigg]. \quad (14)$$

where $s(X_t, \delta_2)$ is the reduced coupon payment in region b.

- superscripts "a" and "b" indicate the normal and bankruptcy (negotiation) regions, respectively.

The reduced coupon payment in region b:

$$s(x, \delta_2) = (1 - \alpha\eta)(1 - \tau)\delta_2 x, \tag{15}$$

- $\eta \in [0, 1]$: firm's bargaining power ($1 - \eta$: bank's bargaining power).

$$\begin{aligned} E_2^{\mathrm{a}}(X_t, c_2, \delta_2) &= \max_{x_2^{\mathrm{d}}} \Pi\delta_2 X_t - (1 - \tau)\frac{c_2}{r} \\ &\quad - \left\{(1 - \alpha\eta)\Pi\delta_2 x_2^{\mathrm{d}} - \frac{c_2}{r}\left(1 - \tau - \tau\frac{\eta\gamma}{\beta - \gamma}\right)\right\}\left(\frac{X_t}{x_2^{\mathrm{d}}}\right)^{\gamma}, \end{aligned} \tag{16}$$

where

$$E_2^{\mathrm{b}}(X_t, c_2, \delta_2) = \eta\left\{\alpha\Pi\delta_2 X_t - \frac{\tau c_2}{r}\frac{\gamma}{\beta - \gamma}\left(\frac{X_t}{x_2^{\mathrm{d}}}\right)^{\beta}\right\}. \tag{17}$$

- $\gamma < 0$ and $\beta > 1$.
- $\{X_t \geq x_2^{\mathrm{d}}\}$: normal region, $\{X_t < x_2^{\mathrm{d}}\}$: bankruptcy region.

Optimal negotiation threshold (maximizing (16) with x_2^{d} gives):

$$x_2^{\mathrm{d}}(c_2, \delta_2) = \underset{x_2^{\mathrm{d}}}{\operatorname{argmax}}\, E_2^{\mathrm{a}}(X_t, c_2, \delta_2) = \kappa_2^{-1} \frac{c_2}{\delta_2}, \tag{18}$$

where

$$\kappa_2 = \frac{\gamma - 1}{\gamma} \frac{1 - \alpha\eta}{1 - \tau(1 - \eta)} \Pi r > 0. \tag{19}$$

Note that

- $x_2^{\mathrm{d}}(c_2, \delta_2)$ is a linear function of c_2.
- $\lim_{c_2 \downarrow 0} x_2^{\mathrm{d}}(c_2, \delta_2) = 0$.
- $\lim_{c_2 \downarrow 0} E_2^{\mathrm{a}}(X_t, c_2, \delta_2) = \Pi \delta_2 X_t$ (due to $\gamma < 0$).
- $\lim_{\eta \downarrow 0} x_2^{\mathrm{d}}(c_2, \delta_2) = x_1^{\mathrm{d}}(c_2, \delta_2)$.
- These results are same as in Mella-Barrel and Perraudin (1997, JF) and Fan and Sundaresan (2000, RFS).

Bank debt value: $D_2^{\mathrm{a}}(X_t, c_2, \delta_2)$

$D_2^{\mathrm{a}}(X_t, c_2, \delta_2)$ and $D_2^{\mathrm{b}}(X_t, c_2, \delta_2)$:

$$\begin{aligned} & D_2^{\mathrm{a}}(X_t, c_2, \delta_2) \\ &= \mathbb{E}_t\left[\int_t^{T_2^{\mathrm{d}}} \mathrm{e}^{-r(u-t)} c_2 \mathrm{d}u + \mathrm{e}^{-r(T_2^{\mathrm{d}}-t)} D_2^{\mathrm{b}}(X_{T_2^{\mathrm{d}}}, c_2, \delta_2)\right], \\ &= \frac{c_2}{r} + (1-\alpha\eta)\Pi\delta_2 x_2^{\mathrm{d}}(c_2, \delta_2)\left(\frac{X_t}{x_2^{\mathrm{d}}(c_2, \delta_2)}\right)^{\gamma} \\ &\quad - \frac{c_2}{r}\left\{1 - \tau + \tau\frac{\beta}{\beta-\gamma} - \tau\frac{\eta\gamma}{\beta-\gamma}\right\}\left(\frac{X_t}{x_2^{\mathrm{d}}(c_2, \delta_2)}\right)^{\gamma}, \end{aligned} \tag{20}$$

where

$$\begin{aligned} & D_2^{\mathrm{b}}(X_t, c_2, \delta_2) \\ &= \mathbb{E}_t\left[\int_t^{T_2^{\mathrm{d}}} \mathrm{e}^{-r(u-t)} s(X_u, \delta_2) \mathrm{d}u + \mathrm{e}^{-r(T_2^{\mathrm{d}}-t)} D_2^{\mathrm{a}}(X_{T_2^{\mathrm{d}}}, c_2, \delta_2)\right] \\ &= (1-\alpha\eta)\Pi\delta_2 X_t - (1-\eta)\frac{\tau c_2}{r}\frac{\gamma}{\beta-\gamma}\left(\frac{X_t}{x_2^{\mathrm{d}}(c_2, \delta_2)}\right)^{\beta}, \end{aligned} \tag{21}$$

Total firm value with bank debt: $V_2^{\mathrm{a}}(X_t, c_2, \delta_2)$

For $X_t > x_2^{\mathrm{d}}(c_2)$,

$$\begin{aligned} \underbrace{V_2^{\mathrm{a}}(X_t, c_2, \delta_2)}_{\text{Levered total firm value}} &= E_2^{\mathrm{a}}(X_t, c_2, \delta_2) + D_2^{\mathrm{a}}(X_t, c_2, \delta_2) \\ &= \underbrace{\Pi\delta_2 X_t}_{\text{unlevered value}} + \underbrace{\frac{\tau c_2}{r}\left\{1 - \frac{\beta}{\beta-\gamma}\left(\frac{X_t}{x_2^{\mathrm{d}}(c_2, \delta_2)}\right)^{\gamma}\right\}}_{\text{tax benefit}} \quad (22) \end{aligned}$$

Note that

- Total firm value in (22) has two components:
  There is no term of bankruptcy costs in (22).
- $\lim_{c_2 \downarrow 0} V_2^{\mathrm{a}}(X_t, c_2, \delta_2) = \lim_{c_2 \downarrow 0} E_2^{\mathrm{a}}(X_t, c_2, \delta_2) = \Pi\delta_2 X_t$
  (due to $\lim_{c_2 \downarrow 0} x_2^{\mathrm{d}}(c_2, \delta_2) = 0$ and $\gamma < 0$).

(P1) Problem for unlevered (all-equity financed) firm

$$E_0^*(x) = \sup_{T_0^{\mathrm{i}}, \delta_0} \mathbb{E}\left[\mathrm{e}^{-rT_0^{\mathrm{i}}} \left\{ E_j^{\mathrm{a}}(X_{T_0^{\mathrm{i}}}, 0, \delta_0) - I(\delta_0) \right\} \right], \tag{23}$$

- $E_j^{\mathrm{a}}(X_{T_j^{\mathrm{i}}}, 0, \delta_j) = E_0^{\mathrm{a}}(X_{T_0^{\mathrm{i}}}, \delta_0) = \Pi \delta_0 X_{T_0^{\mathrm{i}}}$.
- subscript "0" represents the unlevered (all-equity financed) firm.
- superscript "$*$" indicates the optimum without constraint.

(P1): McDonald and Siegel (1986, QJE):

$$E_0^*(x) = \max_{x_0^{\mathrm{i}}, \delta_0} \left(\frac{x}{x_0^{\mathrm{i}}} \right)^{\beta} \left\{ \Pi \delta_0 x_0^{\mathrm{i}} - I(\delta_0) \right\}, \qquad x < x_0^{\mathrm{i}}. \tag{24}$$

Lemma 1 (McDonald and Siegel, 1986, QJE)

Investment volume δ_0^* is given by δ^* satisfying

$$\theta := \frac{\beta}{\beta - 1} = \frac{\delta^* I'(\delta^*)}{I(\delta^*)}. \tag{25}$$

Investment threshold:

$$x_0^{\mathrm{i}*} = \frac{\theta}{\Pi} \frac{I(\delta^*)}{\delta^*}. \tag{26}$$

Equity option value before investment:

$$E_0^*(x) = \left(\frac{x}{x_0^{\mathrm{i}*}}\right)^{\beta} (\theta - 1) I(\delta^*). \tag{27}$$

(P2) Problem for non-constrained levered firm

(P2): Sundaresan and Wang (2007, AER)

$$E^*(x) = \max\{E_1^*(x), E_2^*(x)\}, \tag{28}$$

where $x < \min\{x_1^{\mathrm{i}}, x_2^{\mathrm{i}}\}$ and

$$E_j^*(x) = \max_{x_j^{\mathrm{i}} \geq 0, c_j \geq 0, \delta_j \geq 0} \left(\frac{x}{x_j^{\mathrm{i}}}\right)^{\beta} \left\{ E_j^{\mathrm{a}}(x_j^{\mathrm{i}}, c_j, \delta_j) - \left(I(\delta_j) - D_j^{\mathrm{a}}(x_j^{\mathrm{i}}, c_j, \delta_j) \right) \right\}. \tag{29}$$

where $j \in \{1, 2\}$.

Lemma 2 (Sundaresan and Wang, 2007, AER)

Solutions: δ_j^* is given by δ^* and

$$x_j^{\mathrm{i}*} = \psi_j x_0^{\mathrm{i}*}, \quad c_j^* = \frac{\kappa_j}{h_j} x_j^{\mathrm{i}*}, \quad x_j^{\mathrm{d}*} = \frac{1}{h_j} x_j^{\mathrm{i}*}, \tag{30}$$

where $j \in \{1, 2\}$ and

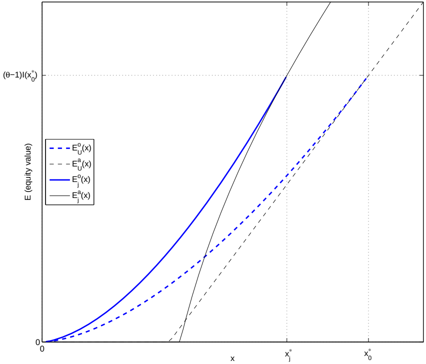
$$h_1 := \left(1 - \gamma\left(1 + \alpha\frac{1-\tau}{\tau}\right)\right)^{-1/\gamma} \geq 1, \quad \psi_1 := \left(1 + \frac{\tau}{1-\tau}\frac{1}{h_1}\right)^{-1} \leq 1,$$
$$h_2 := \left(\frac{\beta}{\beta-\gamma}(1-\gamma)\right)^{-1/\gamma} \geq 1, \quad \psi_2 := \left(1 + \frac{\tau(1-\alpha\eta)}{1-\tau(1-\eta)}\frac{1}{h_2}\right)^{-1} \leq 1.$$

Value:

$$E_j^*(x) = \psi_j^{-\beta} E_0^*(x) = \left(\frac{x}{x_j^{\mathrm{i}*}}\right)^{\beta} (\theta - 1) I(\delta^*), \tag{31}$$

where $j \in \{1, 2\}$.

Relationship between (P1) and (P2)



Investment thresholds:

$x_j^{i*} = \psi_j x_0^{i*}$ where $\psi_j \leq 1$

Equity option values:

$$E_0^*(x) = \left(\frac{x}{x_0^{i*}}\right)^{\beta}(\theta - 1)I(\delta^*)$$

$$E_j^*(x) = \left(\frac{x}{x_j^{i*}}\right)^{\beta}(\theta - 1)I(\delta^*)$$

Values at the investment:

$E_0^*(x_0^{i*}) = E_j^*(x_j^{i*})$

Corollary 3 (Sundaresan and Wang, 2007, AER)

$$x_j^{i*} \leq x_0^{i*}, \quad E_j^*(x) \geq E_0^*(x), \quad E_j^*(x_j^{i*}) = E_0^*(x_0^{i*}), \quad j \in \{1, 2\}. \qquad (32)$$

Corollary 4 (Sundaresan and Wang, 2007, AER)

If $\psi_1 \leq \psi_2$, then we obtain

$$x_1^{\mathrm{i}*} \leq x_2^{\mathrm{i}*} (\leq x_{\mathrm{U}}^{\mathrm{i}*}), \quad E^*(x) = E_1^*(x) \geq E_2^*(x) \geq E_0^*(x).$$

Otherwise ($\psi_1 > \psi_2$), we have

$$x_2^{\mathrm{i}*} < x_1^{\mathrm{i}*} (\leq x_{\mathrm{U}}^{\mathrm{i}*}), \quad E^*(x) = E_2^*(x) > E_1^*(x) \geq E_0^*(x).$$

Definition (Symmetric relationship)

Under no financing constraints, we have symmetric relationship, i.e.,

$$x_1^{\mathrm{i}*} \leq x_2^{\mathrm{i}*} \text{ iff } E_1^*(x) \geq E_2^*(x).$$

(P3) Problem for constrained levered firm

(P3): Our original model

$$E^{**}(x) = \max\{E_1^{**}(x), E_2^{**}(x)\}, \tag{33}$$

where $x < \min\{x_1^{\mathrm{i}}, x_2^{\mathrm{i}}\}$ and

$$E_j^{**}(x) = \max_{x_j^{\mathrm{i}}\geq 0, c_j \geq 0, \delta_j \geq 0} \left(\frac{x}{x_j^{\mathrm{i}}}\right)^{\beta} \{E_j^{\mathrm{a}}(x_j^{\mathrm{i}}, c_j, \delta_j) - (I(\delta_j) - D_j^{\mathrm{a}}(x_j^{\mathrm{i}}, c_j, \delta_j))\},$$

$$\text{subject to } \frac{D_j^{\mathrm{a}}(x_j, c_j, \delta_j)}{I(\delta_j)} \leq q, \tag{34}$$

where $q \geq 0$ and $j \in \{1, 2\}$.

- superscript "$**$" represents the optimum with constraint
- our problem (P3) includes two problems (P1) and (P2).
  - When $q = 0$, (P3) becomes (P1).
  - When q is a sufficient large, (P3) turns out to be (P2).

Our conjecture about the solution to (P3)

Intuitively conjecture

- Intuitively conjecture 1:

$$\underbrace{x_j^{i*}}_{(q\uparrow+\infty)} \le \underbrace{x_j^{i**}}_{(q<+\infty)} \le \underbrace{x_0^{i*}}_{(q\downarrow 0)}, \quad j \in \{1,2\}. \tag{35}$$

  where "$**$" stands for the optimum of the constrained problem.

- Intuitively conjecture 2:

$$x_1^{i**} \le x_2^{i**} \text{ iff } E_1^{**}(x) \ge E_2^{**}(x). \tag{36}$$

But, these conjectures are not always correct...

Critical value that the firm is financially constrained

Lemma 5

Given a debt structure j ($j \in \{1,2\}$), we have $\delta_j^{**} = \delta^*$. We define a critical value x_j^{p} by

$$\frac{D_j^{\mathrm{a}}(x_j^{\mathrm{p}}, c_j(x_j^{\mathrm{p}}, \delta^*), \delta^*)}{I(\delta^*)} = q \geq 0, \tag{37}$$

where δ^* satisfying $\theta = \delta^* I'(\delta^*)/I(\delta^*)$ and

$$c_j(x, \delta^*) \quad := \quad \operatorname*{argmax}_{c_j} V_j^{\mathrm{a}}(x, c_j, \delta^*) = \frac{\kappa_j}{h_j}\delta^* x, \tag{38}$$

Then there exists a unique critical value x_j^{p}.

Here, (38) is shown by Leland (1994, JF). The left hand side of (37), $D_j^{\mathrm{a}}(x, c_j(x, \delta^*), \delta^*)/I(\delta^*)$, is strictly monotonically increasing continuous function of x with $\lim_{x \downarrow 0} D_j^{\mathrm{a}}(x, c_j(x, \delta^*), \delta^*)/I(\delta^*) = 0$ and $\lim_{x \uparrow +\infty} D_j^{\mathrm{a}}(x, c_j(x, \delta^*), \delta^*)/I(\delta^*) = +\infty$.

Solution and numerical examples

Lemma 6

If $x_j^{\mathrm{i}*} > x_j^{\mathrm{p}}$ $(x_j^{\mathrm{i}*} \leq x_j^{\mathrm{p}})$, the firm is (not) financially constrained.

Lemma 7

If $x_j^{\mathrm{i}*} \leq x_j^{\mathrm{p}}$, we have $x_j^{\mathrm{i}**} = x_j^{\mathrm{i}*}$ and $c_j^{**} = c_j^{*}$. Otherwise $(x_j^{\mathrm{i}*} > x_j^{\mathrm{p}})$, the solutions $(x_j^{\mathrm{i}**}, c_j^{**})$ are obtained by solving....

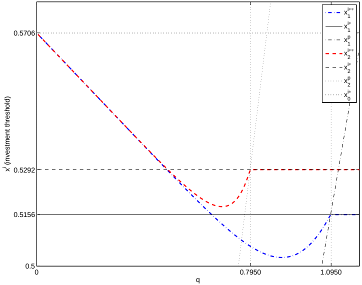
- $**$ represents the solution to the constrained problem.
- We cannot solve the solution analytically. However, we consider the properties of the solution analytically.
- cost function

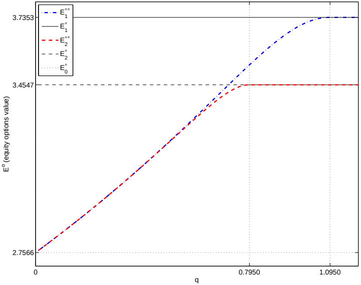
$$I(\delta_j) = \delta_0 + \delta_j^2, \quad \delta_0 > 0, \tag{39}$$

where $j \in \{1, 2\}$
- Basic parameters

$$r = 0.09; \mu = 0.02; \delta_0 = 4; \tau = 0.15; \alpha = 0.4; X_0 = x = 0.4.$$

Thresholds and values for $\eta = 1$ and $\sigma = 0.1$



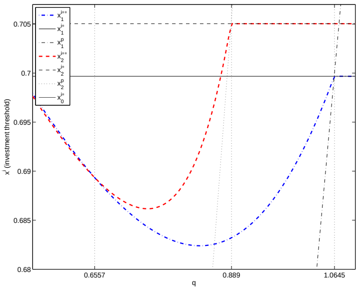


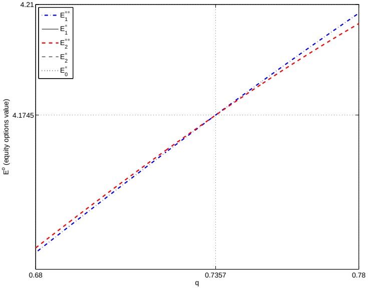
- If $q < 1.0950$ ($q < 0.7950$), market (bank) debt issuance limit constraint is binding.
- Investment thresholds are non-monotonic with q.
  Conjecture 1, $x_j^{i*} \leq x_j^{i**} \leq x_0^{i*}$, is not always correct.

Observation 1

For $\sigma = 0.1$, the firm prefers market to bank debt financing for any q.

Thresholds and values for $\eta = 1$ and $\sigma = 0.15$

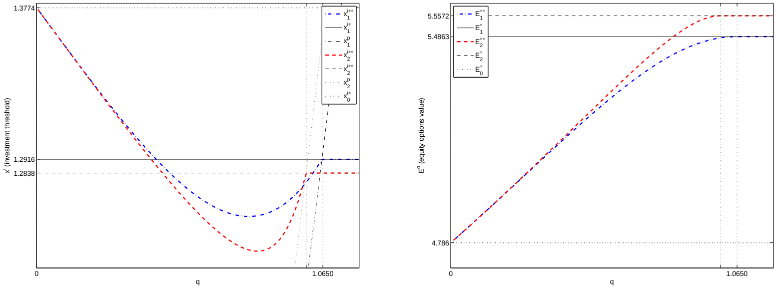




Observation 2

For $\sigma = 0.15$, there exits a unique $\hat{q} = 0.7357$.
If $q \geq \hat{q}$, the firm chooses market debt issuance.
Otherwise ($q < \hat{q}$), the firm does bank debt issuance.

Thresholds and values for $\eta = 1$ and $\sigma = 0.2$



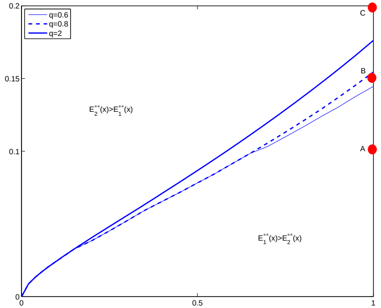
Observation 3

For $\sigma = 0.2$, the firm prefers bank to market debt financing for any q.

Regions of $E^{o}_{1C}(x) \geq E^{o}_{2C}(x)$ in (η, σ) space

Observation 4

Which of bank and market debt is preferred depends on three key parameters: q (friction), σ (volatility), and η (bargaining power).



Observation 5

An increase in debt issuance limit is more likely to issue market debt.

Bank debt or Market debt?

Definition (large/mature firm)

- The firms with larger q, smaller σ, and larger η best approximate large/mature corporations
- The firms with smaller q larger σ and smaller η best approximate small/young corporations

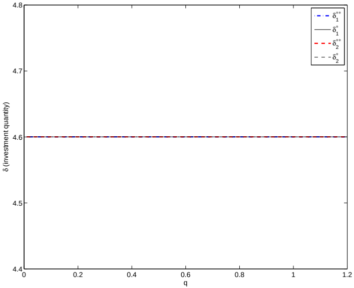
The definition is based on Rajan (1992, JF) and Hackbarth et al. (2007, RFS).

Observation 6

- Small/young firms are more likely to issue bank debt.
- Large/mature firms are more likely to issue market debt.

There results fit well with stylized facts proposed by Bolton and Freixas (2000, JPE) and empirical studies by Blackwell and Kidwell (1988, JFE), Cantillo and Wright (2000, RFS), and Denis and Mihov (2003, JFE).

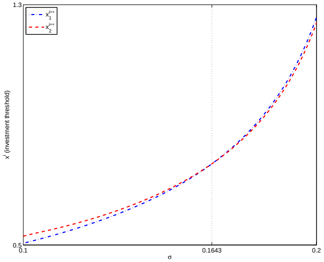
Invariance of investment quantity with frictions

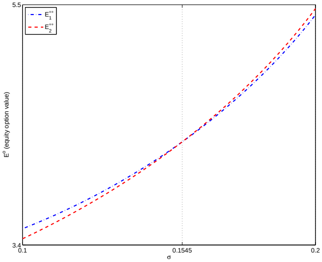


Observation 7

Investment quantity $\delta^* = \delta_1^{**} = \delta_2^{**}$ is invariant with frictions.

No symmetric relationship





Suppose $q \uparrow +\infty$. Recall that $x_1^{i*} \le x_2^{i*}$ iff $E_1^*(x) \ge E_2^*(x)$.
In contrast, suppose, e.g., $q = 0.8 < +\infty$:

- $x_1^{i**} \le x_2^{i**}$ for $\sigma \in [0.1, 0.1643]$.
- $E_1^{**}(x) \ge E_2^{**}(x)$ for $\sigma \in [0.1, 0.1545]$.
- For $\sigma \in [0.1545, 0.1643]$, $x_1^{i**} \le x_2^{i**}$ and $E_1^{**}(x) < E_2^{**}(x)$.

Observation 9

Under constraints, symmetric relationship is not necessarily obtained.

Implied results of asymmetric relationship

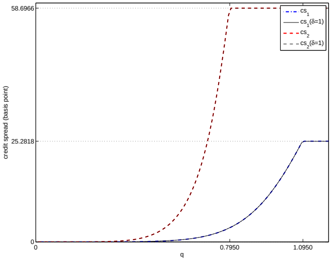
Implication under no financing friction

When the firm has an option of choosing debt structure, the investment threshold is *always decreased* and the equity value is *always increased*.

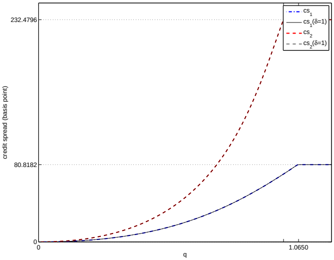
Implication under financing friction

When the firm has an option of choosing debt structure, the investment threshold is *not always decreased* although the equity value is *always increased*.

Credit spreads: $cs_j := c_j / D_j^{\mathrm{a}}(x_j^{\mathrm{i}}, c_j) - r$



$\sigma = 0.1$ (market debt is preferred)

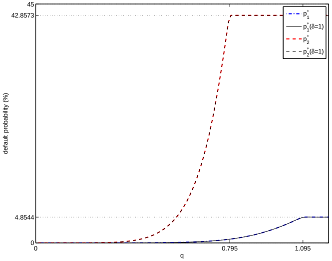


$\sigma = 0.2$ (bank debt is preferred)

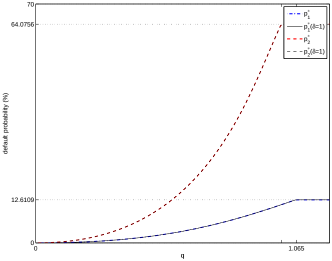
Observation 10

Financial constraints lead to a decrease in credit spreads.
Credit spreads for bank debt are always larger those for market debt.

Default probabilities: $p_j := (x_j^{\mathrm{i}}/x_j^{\mathrm{d}})^\gamma$



$\sigma = 0.1$ (market debt is preferred)



$\sigma = 0.2$ (bank debt is preferred)

Observation 11

Financial constraints lead to a decrease in default probabilities. Default probabilities for bank debt are always larger than those for market debt.

Constraint leads to low-risk and low-return

Definition (Risk and return to debt holders)

The credit spreads and default probabilities can be regarded as the return and risk of debt holders.

Observation 12

Financing constraints lead to low-risk and low-return of debt holders.

Observation 13

Bank debt is high-risk and high-return, compared with market debt.

These results are consistent with empirical studies (e.g., Davydenko and Strebulaev, 2007, JF).

Concluding remarks

1. Our contribution is to introduce debt issuance limit constraint along with bank and market debt and thus to provide the first synthesis of investment, financing, and debt structure choice theories.
2. Four novel results:
   - An increase in capacity q is more likely to issue market debt.
   - $x_j^{i*} \leq x_j^{i**} \leq x_{\mathrm{U}}^{i*}$ is not always correct,
     i.e., investment threshold is not monotonic with q
     - x_j^{i**} has a U-shaped curve with q for any j ($j \in \{1, 2\}$).
   - $x_1^{i**} \leq x_2^{i**}$ iff $E_1^{**}(x) \geq E_2^{**}(x)$ is not always correct.
     - This result is contrary to that under no financing constraints.
   - Financing constraints lead to a decrease in credit spreads and default probabilities, i.e., low-risk and low-return.
     - Bank debt is high-risk and high-return, compared with market debt.
3. Our theoretical results are consistent with stylized facts and empirical results.