Optional splitting formula in a progressively enlarged filtration

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Summary

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The subject of our talk concerns the theme of the enlargement of filtrations. By definition, an enlargement of filtration is a problem of measurability. But, thanks to the genius of the pioneers, this theory has become a theory of computation, more structural and more operational. This sometimes makes us lose the attention on the issue of measurability.

Summary

Let $\mathbb{F}$ be a filtration and τ be a random time. Let $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ be the progressive enlargement of $\mathbb{F}$ with τ defined by

$$\mathcal{G}_t = \mathcal{N}^{\sigma(\tau) \vee \mathcal{F}_\infty} \vee (\cap_{s>t}(\mathcal{F}_s \vee \sigma(\tau \wedge s))), \ \ t \geq 0.$$

This family $\mathbb{G}$ is a right-continuous filtration.

We study the optional splitting formula in $\mathbb{G}$. We are interested in this formula because of

1. its fundamental role in many recent papers on credit risk modeling,
2. its validity is limited in scope,
3. this limitation is not sufficiently underlined,
4. it is associated with some interesting measurability properties.

Content

Some examples of the measurability problem

The word "measurability" denotes a relation a random map has with respect to a σ-algebra.

We begin with Barlow's paper, which is one of the first work on enlargement of filtration theory. A honest time τ is considered in this paper. It is shown that the $\mathbb{G}$-progressively measurable processes can be written in term of the random intervals $[0,\tau),[\tau,\infty)$ and of the $\mathbb{F}$-progressively measurable processes.

Some examples of the measurability problem

Similarly, for the $\mathbb{G}$-predictable processes, they satisfy the formula :

For any $Y \in \mathcal{P}(\mathbb{G})$, there exist $Y', Y'' \in \mathcal{P}(\mathbb{F})$ such that

$$Y \mathbb{1}_{(0,\infty)} = Y' \mathbb{1}_{(0,\tau]} + Y'' \mathbb{1}_{(\tau,\infty)}. \tag{1}$$

This relationships is one of the elements which imply that every $\mathbb{F}$-martingale is a $\mathbb{G}$-semimartingale in the case of a honest time τ.

Some examples of the measurability problem

If τ is not a honest time, we have a less precise formula proved in Jeulin's book: for any $Y \in \mathcal{P}(\mathbb{G})$, there exist a $\mathbb{F}$-predictable process Y' and a function Y'' defined on $[0, \infty] \times (\mathbb{R}_+ \times \Omega)$ being $\mathcal{B}[0, \infty] \otimes \mathcal{P}(\mathbb{F})$ measurable, such that

$$Y = Y' \mathbb{1}_{[0,\tau]} + Y''(\tau) \mathbb{1}_{(\tau,\infty)}. \quad (2)$$

In particular, $[0, \tau] \cap \mathcal{P}(\mathbb{G}) = [0, \tau] \cap \mathcal{P}(\mathbb{F})$. This formula is used in various computations in the filtration $\mathbb{G}$ which vary from the predictable dual projections to the orthogonal decomposition of the family of $\mathbb{G}$-martingales stopped at τ.

Some examples of the measurability problem

In the paper of Kusuoka, the martingale representation property in $\mathbb{G}$ is studied for a Brownian filtration $\mathbb{F}$ and a random time τ satisfying the two conditions:

(i) any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale (called hypothesis(H)), and

(ii) the σ-algebras $\mathcal{G}_t^\circ = \sigma(\tau \wedge t) \vee \mathcal{F}_t, t \geq 0$, completed by the null sets, form a right-continuous filtration.

The condition (ii) is a measurability condition and it is not trivial. We remark already that $\{\tau = t\} \notin \mathcal{G}_t^\circ$, but always $\{\tau = t\} \in \mathcal{G}_{t+}^\circ$.

Some examples of the measurability problem

The paper of Bélanger-Shreve-Wong considers another filtration $\mathcal{G}_t^\star = \sigma(\{\tau \leq s\}: 0 \leq s \leq t) \vee \mathcal{F}_t, t \geq 0$. The filtration $\mathbb{F}$ is supposed to be a complete Brownian filtration and the random time τ to be a Cox time, i.e.

$$\tau = \inf\{t \geq 0: \Gamma_t \geq \Xi\},$$

where Γ is a $\mathbb{F}$-adapted càdlàg increasing process and Ξ is a strictly positive random variable independent of $\mathcal{F}_\infty$. Then, it is proved that $(\mathcal{G}_t^\star)_{t\geq 0}$ is a right-continuous filtration, and consequently $\mathcal{G}_t = \mathcal{G}_t^\star$.

Some examples of the measurability problem

Bélanger-Shreve-Wong's result is a typical example of the problem studied in Weizsacker: in what circumstances does the following formula hold:

$$\mathcal{T}' \vee (\cap_{n=1}^{\infty} \mathcal{T}_n) = \cap_{n=1}^{\infty} (\mathcal{T}' \vee \mathcal{T}_n),$$

where $\mathcal{T}'$ is a σ-algebra and $(\mathcal{T}_n)_{n \geq 1}$ is an inverse filtration. This interchangeability problem is in general a very delicate issue. See for example the comments of Handel to have some idea about that.

Some examples of the measurability problem

This result of Bélanger-Shreve-Wong also is a particular case of the following question: how can the σ-algebra $\mathcal{G}_T$, where T is a $\mathbb{F}$-stopping time, be factorized in terms of $\sigma(\{\tau \le s \wedge T \colon s \ge 0\})$ and of $\mathbb{F}_T$.

Many works on $\mathbb{G}$ depends on that decomposition, especially when the monotone class theorem is applied on $\mathbb{G}_T$. For example, we have the identity $\mathcal{G}_\infty = \sigma(\tau) \vee \mathcal{F}_\infty$ (completed by the null sets). This is required in the paper of Kusuoka in order to obtain results on the martingale representation property in $\mathbb{G}$ under the hypothesis(H).

Some examples of the measurability problem

When the results of Kusuoka are extended in Jeanblanc-Song, one has to work with a general $\mathbb{F}$-stopping time T other than the ∞. But usually the σ-algebra $\mathcal{G}_T$ is strictly greater than $\sigma(\{\tau \le s \wedge T \colon s \ge 0\}) \vee \mathcal{F}_T$. A laborious computation was necessary in Jeanblanc-Song to get around the gap between them.

To better appreciate this idea, it is to be compared with the general equality $\mathcal{G}_{T-} = \sigma(\tau \wedge T) \vee \mathcal{F}_{T-}$ (completed with null sets), a consequence of formula (2) and of the identity $\{T \le \tau\} = \{T = \tau \wedge T\}$.

Some examples of the measurability problem

In other respects, the work of Biagini-Cretarola requires the following fact: for a complete Brownian filtration $\mathbb{F}$, for a random time τ whose hazard process is continuous and increasing, for any $\mathbb{G}$-martingale X, there exists a $\mathbb{F}$-predictable process J such that $X_\tau = J_\tau$ on $\{\tau < \infty\}$. This is equivalent to say $\{\tau < \infty\} \cap \mathcal{G}_\tau = \{\tau < \infty\} \cap \mathcal{G}_{\tau-}$.

In general these two σ-algebras are different. The gap between such σ-algebras was the subject of several papers in the literature. Biagini-Cretarola's result was obtained by a direct computation. It is useful to see if the result is consequence of some more general principle.

Optional splitting formula for one random time

Recently an optional version of formula (2) has been revealed to be fundamental in credit risk modeling with progressive enlargement of filtration: for any $\mathbb{G}$-optional process Y, there exist a $\mathbb{F}$-optional process Y' and a function Y'' defined on $[0,\infty] \times (\mathbb{R}_+ \times \Omega)$ being $\mathcal{B}[0,\infty] \otimes \mathcal{O}(\mathbb{F})$ measurable, such that

$$Y = Y' \mathbb{1}_{[0,\tau)} + Y''(\tau) \mathbb{1}_{[\tau,\infty)}. \tag{3}$$

We will call that formula optional splitting formula (in comparison with the predictable splitting formula (2)). The term "splitting" is twofold. It obviously means that the formula is splitted at the random time τ. But, more importantly, it underlines that the measurability of $Y''(\tau)$ is factorized into two components $\sigma(\tau)$ and $\mathcal{O}(\mathbb{F})$ ($Y'' \in \mathcal{B}[0,\infty] \otimes \mathcal{O}(\mathbb{F})$).

Optional splitting formula for one random time

This formula (3) has been directly or indirectly involved in numerous works. That said, this widespread use of the formula suggests caution. In fact, unlike formula (2), formula (3) is in general not valid. We recall the well-known example of Barlow: let $\mathbb{F}$ be the natural filtration of a Brownian motion B with $B_0 = 0$. Let $T = \inf\{t \geq 0 \colon |B_t| = 1\}$ and $\tau = \sup\{s \leq T \colon B_s = 0\}$. Then, $X = \mathbb{1}_{[\tau,\infty)} \mathrm{sign}(B_T)$ is a $\mathbb{G}$-martingale. If this process X satisfied the optional splitting formula, the process Y'' could be chosen $\mathbb{F}$-predictable. Consequently $\Delta_\tau X \in \mathcal{G}_{\tau-}$ which contradicts the martingale property of X.

See also Jeulin's book who extends this example to a general theorem.

Optional splitting formula for one random time

Knowing that counter-example, we wondered if the use of the optional splitting formula in the literature was justified. To have an answer, we had followed the usual stages : We looked first the consequences of such a formula. We then tried to find the sufficient conditions. Moreover, since many works in the literature used a multi-times version of the optional splitting formula, we had included the multi-times version in our study. With these results we had investigated the validity of the use of the optional splitting formula.

Some immediate consequences

consequence on $\mathcal{G}_\tau$

We begin the investigation with a single random time τ.
For any random time R on Ω, we denote

$$\mathcal{F}_R = \sigma\{X_R \mathbb{1}_{\{R<\infty\}} : \ X \text{ an } \mathbb{F}\text{-optional process}\},$$
$$\mathcal{F}_{R+} = \sigma\{X_R \mathbb{1}_{\{R<\infty\}} : \ X \text{ an } \mathbb{F}\text{-progressively measurable process}\}.$$

Theorem. Assume the optional splitting formula at τ. We necessarily have $\mathcal{F}_\tau = \mathcal{F}_{\tau+} = \mathcal{G}_\tau$.

Some immediate consequences

consequence on $\mathcal{G}_\tau$

The above equality of σ-algebras is not valid in general. Azéma-Yor give a systematical construction of τ which does not satisfy the above equality (see Jeulin): Let M be a continuous uniformly integrable $\mathbb{F}$-martingale such that $M_0 = 0, M_\infty \neq 0$. Let $\tau = \sup\{t \geq 0 \colon M_t = 0\}$. Then, $\mathcal{F}_\tau \neq \mathcal{F}_{\tau+}$.

It is a generalization of Barlow's example. Once again it proves that the optional splitting formula at τ can not hold in general.

Some immediate consequences

the factorizability of $\mathcal{G}_t$

For two elements a, b in $[0, \infty]$ we denote

$$a \nmid b = \begin{cases} a & \text{if } a \leq b \\ \infty & \text{if } a > b. \end{cases}$$

Theorem. If the optional splitting formula holds at τ, then for any $t \geq 0$, $\mathcal{G}_t = \mathcal{N} \vee \sigma(\tau \nmid t) \vee \mathcal{F}_t$.

As a consequence of this theorem, the filtration $(\mathcal{N} \vee \sigma(\tau \nmid t) \vee \mathcal{F}_t \colon t \geq 0)$ is right-continuous.

As a matter of fact, in the above theorem we can not replace the term $\tau \nmid t$ with $\tau \wedge t$. In general,

$$\{t < \tau\} \cap (\sigma(\tau \wedge t) \vee \mathcal{F}_t) + \{\tau \leq t\} \cap (\sigma(\tau \wedge t) \vee \mathcal{F}_t) \neq \sigma(\tau \wedge t) \vee \mathcal{F}_t,$$

because $\{\tau \leq t\}$ (or more precisely $\{\tau = t\}$) is not necessarily in $\sigma(\tau \wedge t) \vee \mathcal{F}_t$. We can laugh at this detail. Nevertheless, it is a fault. See [Dellacherie-Meyer Chapitre IV, paragraph 104] which comments on an example in Dellacherie. Also it is better to notice that the problem has been overlooked in many papers in the literature. (The problem no longer arises if $\{\tau = t\}$ is negligible and if $\mathbb{F}$ is complete.)

Theorem. The $\mathbb{G}$-predictable processes satisfy the optional splitting formula.

Optional splitting formulas which hold partially

We notice that the the optional splitting formula problem can not be treated by itself. It should be considered as a particular case of a more broad problem. We consider the family $\mathcal{L}^o$ of $\mathbb{G}$-optional subsets $A \subset \mathbb{R}_+ \times \Omega$ such that, for any $\mathbb{G}$-optional process Y, there exists a $\mathbb{F}$-optional process Y' and a function Y'' defined on $[0, \infty] \times (\mathbb{R}_+ \times \Omega)$ being $\mathcal{B}[0, \infty] \otimes \mathcal{O}(\mathbb{F})$ measurable, such that

$$Y \mathbb{1}_A = (Y' \mathbb{1}_{[0,\tau)} + Y''(\tau) \mathbb{1}_{[\tau,\infty)}) \mathbb{1}_A. \tag{4}$$

We say then that the optional splitting formula at τ holds on A.

Optional splitting formulas which hold partially

Formula (3) is the particular case of formula (4) when $A = \mathbb{R}_+ \times \Omega$. To make the difference, we call formula (3) the global optional splitting formula. The question now becomes whether $\mathbb{R}_+ \times \Omega \in \mathcal{L}^o$, or more generally, exactly which elements are contained in the family $\mathcal{L}^o$.

We note that, no matter if formula (3) holds, the family $\mathcal{L}^o$ always gives good indications of what the filtration $\mathbb{G}$ looks like.

Optional splitting formulas which hold partially

the predictable elements in $\mathcal{L}^o$

Theorem. Let A be a $\mathbb{G}$-predictable set. Then, $A \in \mathcal{L}^o$ if and only if, for any $\mathbb{G}$-optional process Y, $Y\mathbb{1}_A$ satisfies the global optional splitting formula.

Let $(A_i)_{i=1}^{\infty}$ be a sequence of $\mathbb{G}$-predictable sets. Suppose that $(A_i)_{i=1}^{\infty} \subset \mathcal{L}^o$. Then, $\cup_{i=1}^{\infty} A_i \in \mathcal{L}^o$.

We will especially interested in the intervals elements in $\mathcal{L}^o$, because of the sufficient condition we will give later.

Proposition. Let S, T be two $\mathbb{G}$-stopping times. To have the local optional splitting formula on $[S, T)$, it is necessary and sufficient that, for any bounded $(\mathbb{Q}, \mathbb{G})$-martingale X such that $X_T \in \mathcal{G}_{T-}$, X satisfies the optional splitting formula on $[S, T)$.

Proposition. Let R be a $\mathbb{G}$-stopping time. Then, $[R] \in \mathcal{L}^o$, if and only if

$$\{R < \infty\} \cap \mathcal{G}_R = \{R < \infty\} \cap (\mathcal{N} \vee \sigma(\tau \nmid R) \vee \mathcal{F}_R).$$

Optional splitting formulas which hold partially

the intervals elements in $\mathcal{L}^o$

Proposition. Let S, T be $\mathbb{G}$-stopping times. Suppose that $(S, T) \in \mathcal{L}^o$ and $[T_{\{S<T<\infty\}}] \in \mathcal{L}^o$. Suppose that $\mathbb{1}_{[T_{\{S<T<\infty\}}]}$ satisfies the optional splitting formula on $(S, T]$. Then, $(S, T] \in \mathcal{L}^o$.

As expected, we have the following result.
Theorem. $[0, \tau) \in \mathcal{L}^o$.

The proof of this result is easy. Let $\xi \in \mathcal{G}_\infty$ be a bounded random variable. Let us abuse the notation ξ to also denote the bounded martingale $\mathbb{E}^{\mathbb{Q}}[\xi|\mathcal{G}_t], t \geq 0$. We write the identity:

$$\xi_t \mathbb{1}_{\{t<\tau\}} = \mathbb{1}_{\{t<\tau\}} \frac{\mathbb{Q}[\xi \mathbb{1}_{\{t<\tau\}}|\mathcal{F}_t]}{\mathbb{Q}[t < \tau|\mathcal{F}_t]} \mathbb{1}_{\{\mathbb{Q}[t<\tau|\mathcal{F}_t]>0\}}, t \geq 0.$$

This is an optional splitting formula for ξ on $[0, \tau)$. Now, we apply the propositions on intervals elements to conclude.

As usual, there exists no complete solution for the situation of the interval $[\tau, \infty)$. The following notion induces a sufficient condition which will be stroug enough to meet the pratical purposes.

Notion. Let S, T be $\mathbb{G}$-stopping times. A probability measure $\mathbb{Q}'$ defined on $\mathcal{G}_\infty$ is called an $\mathcal{sH}$-measure over the random time interval $(S, T]$ (with respect to $(\mathbb{Q}, \mathbb{F}, \mathbb{G})$), if $\mathbb{Q}'$ is equivalent to $\mathbb{Q}$ on $\mathcal{G}_\infty$, and if, for any $(\mathbb{Q}, \mathbb{F})$ local martingale X, $X^{(S,T]}$ is a $(\mathbb{Q}', \mathbb{G})$ local martingale, where $X_t^{(S,T]} = X_t^{S \vee T} - X_t^S, t \geq 0$.

Proposition. For any $\mathbb{F}$-stopping time T, for any $\mathbb{G}$-stopping time S such that $S \geq \tau$ (an almost sure relation), if an $s\mathcal{H}$-measure $\mathbb{Q}'$ over $(S, T]$ exists, then $[S, T) \in \mathcal{L}^o$.

Proposition. Let R be a $\mathbb{G}$-stopping time. For any $\mathbb{F}$-stopping time T and any $\mathbb{G}$-stopping time S, if an $s\mathcal{H}$-measure $\mathbb{Q}'$ over $(S, T]$ exists, we have

$$\{\tau \leq R\} \cap \{S \leq R < T\} \cap \mathcal{G}_R = \{\tau \leq R\} \cap \{S \leq R < T\} \cap (\mathcal{N} \vee \sigma(\tau) \vee \mathcal{F}_R).$$

Theorem. Suppose that there exists a countable family of pairs of $\mathbb{G}$ stopping times $\{S_j, T_j\}, j \in \mathbb{N}$, such that

(1) T_j are $\mathbb{F}$-stopping times ;

(2) $(\tau, \infty) \subset \cup_{i \in \mathbb{N}}(S_j, T_j)$ (covering condition on (τ, ∞)).

Suppose that, for any $j \in \mathbb{N}$, there exists an $\mathscr{sH}$-measure $\mathbb{Q}_j$ over the time interval $(S_j, T_j]$. Then $(\tau, \infty) \in \mathcal{L}^o$.
If we replace the condition (2) with the condition:

(2)' $[\tau, \infty) \cap (0, \infty) \subset \cup_{i \in \mathbb{N}}(S_j, T_j)$ (the covering condition on $[\tau, \infty)$).

Then, the global optional splitting formula holds.

Despite its unusual definition, the $\mathcal{SH}$-measure condition is satisfied in most of examples we know in the literature. We begin with the following result.

We say that hypothesis(H) is satisfied between the pair of filtrations $(\mathbb{F}, \mathbb{G})$ under $\mathbb{Q}$, if every $(\mathbb{Q}, \mathbb{F})$-martingale is a $(\mathbb{Q}, \mathbb{G})$-martingale. The hypothesis(H) is satisfied if τ is independent of $\mathcal{F}_\infty$ or if τ is a Cox time.
Theorem. If there exists a probability measure $\mathbb{Q}'$ equivalent to $\mathbb{Q}$ such that hypothesis(H) is satisfied under $\mathbb{Q}'$, then the probability measure $\mathbb{Q}'$ is an $\mathcal{SH}$-measure over $(0, \infty]$. Consequently, the global optional splitting formula holds.

The work of Bélanger-Shreve-Wong raises the problem of establishing the right-continuity of the filtration of $\sigma(\tau \nmid t) \vee \mathcal{F}_t$ (completed by the null sets), $t \geq 0$, when τ is a Cox time. We know that a Cox time satisfies hypothesis(H). According to the above theorem of hypothesis(H), the global optional splitting formula holds. Applying the theorem of factorizability we obtain $\mathcal{G}_t = \mathcal{N} \vee \sigma(\tau \nmid t) \vee \mathcal{F}_t, t \geq 0$. The result of Bélanger-Shreve-Wong is proved, because $\mathbb{G}$ is a right-continuous filtration.

The question raised in Biagini-Cretarola is to find, for any $\mathbb{G}$-martingale Z, a $\mathbb{F}$-predictable process $\hat{Z}$ such that $Z_\tau = \hat{Z}_\tau$ if $\tau < \infty$. This is equivalent to saying that $\{\tau < \infty\} \cap \mathcal{G}_\tau = \{\tau < \infty\} \cap \mathcal{G}_{\tau-}$. Parallel to the result of Biagini-Cretarola, this question can also be treated with the optional splitting formula. Indeed, if we suppose that τ is a Cox time. Then the global optional splitting formula holds, which implies that $[\tau] \in \mathcal{L}^o$, according to the theorem of $[R]$,

$$\{\tau < \infty\} \cap \mathcal{G}_\tau = \{\tau < \infty\} \cap (\mathcal{N} \vee \sigma(\tau) \vee \mathcal{F}_\tau).$$

If $\mathbb{F}$ is moreover a Brownian filtration,

$$\mathcal{N} \vee \sigma(\tau) \vee \mathcal{F}_\tau = \mathcal{N} \vee \sigma(\tau) \vee \mathcal{F}_{\tau-} = \mathcal{G}_{\tau-},$$

which yields the desired equality.

Examples

a Kusuoka's paper

In Kusuoka the author works with a random time τ whose probability distribution is continuous and which satisfies Hypothesis(H). It is also assumed that the filtration $\sigma(\tau \wedge t) \vee \mathcal{F}_t, t \geq 0$, is right-continuous. Let us show that there is no need to assume this right-continuity, because it is the consequence of the other assumptions. Actually, since τ has a continuous distribution,

$$\mathcal{N} \vee \sigma(\tau \wedge t) \vee \mathcal{F}_t = \mathcal{N} \vee \sigma(\tau \nmid t) \vee \mathcal{F}_t$$

Now applying the theorem of factorizability (passing through the theorem of Hypothesis(H)), $\mathcal{N} \vee \sigma(\tau \nmid t) \vee \mathcal{F}_t$ coincides with $\mathcal{G}_t$, which is right-continuous.

A random time is called honest if it is equal to the end of an optional set, when it is finite. The honest time is a good notion to modelize the bankrupt. According to Barlow and Jeulin, in general the model with a honest time does not satisfy the optional splitting formula. The problem mainly comes from the difference between $\mathcal{G}_{\tau-}$ and $\mathcal{G}_{\tau}$. Jeulin gives some technique to determine if the difference exists. In the paper of Barlow-Emery-Knight-Song-Yor, it is proved that, when $\mathbb{F}$ is a Brownian motion filtration, this difference is at most of one bit. In general, we do not know how to determine this difference. That said, using the theorem of $\mathscr{s}\mathcal{H}$-measure with covering condition, it can be proved that $(\tau, \infty) \in \mathcal{L}^o$ for a honest time τ in a Brownian filtration.

We present an example, developped in Jeanblanc-Song that we called the ($\natural$)-model, where the theorem of hypothesis(H) can not be applied, but the optional splitting formula holds.

We consider an $\mathbb{F}$-adapted continuous increasing process Λ and a positive $\mathbb{F}$-local martingale N. We suppose that $\Lambda_0 = 0, N_0 = 1$ and $0 \leq N_t e^{-\Lambda_t} \leq 1$ for all $0 \leq t < \infty$.

Theorem. Suppose **Hy**(C), i.e. all $(\mathbb{Q}, \mathbb{F})$ local martingales are continuous. Suppose $0 < Z_t < 1$ for any $0 < t < \infty$, where $Z = Ne^{-\Lambda}$. Then, for any $(\mathbb{Q}, \mathbb{F})$ local martingale Y, for any bounded differentiable function f with bounded continuous derivative and $f(0) = 0$, there exists a probability measure $\mathbb{Q}^\natural$ and a random time τ (defined on an extension of the basic probability space) such that, for any $u \in \mathbb{R}_+^*$, the martingale $M_t^u = \mathbb{Q}^\natural[\tau \leq u | \mathcal{F}_t], t \geq u$, satisfies the following evolution equation(♮):

$$(\natural_u) \left\{ \begin{array}{l} dX_t = X_t \left(-\frac{e^{-\Lambda_t}}{1-Z_t} dN_t + f(X_t - (1 - Z_t)) dY_t \right), \ u \leq t < \infty \\ X_u = 1 - Z_u. \end{array} \right.$$

Examples

♮-model

Under the additional assumption:
Hy(Mc): For each $0 < t < \infty$, the map $u \to M_t^u$ is continuous on $(0, t]$,
it is also proved

Theorem. For any $(\mathbb{Q}^\natural, \mathbb{F})$ local martingale X, the process

$$\Gamma(X)_t = \int_0^t 1\!\!1_{\{s \leq \tau\}} \frac{e^{-\Lambda_s}}{Z_s} d\langle N, X\rangle_s - \int_0^t 1\!\!1_{\{\tau < s\}} \frac{e^{-\Lambda_s}}{1 - Z_s} d\langle N, X\rangle_s$$

$$+ \int_0^t 1\!\!1_{\{\tau < s\}} (f(M_s^\tau - (1 - Z_s)) + M_s^\tau f'(M_s^\tau - (1 - Z_s))) d\langle Y, X \qquad (5)$$

$0 \leq t < \infty$, is a well-defined $\mathbb{G}$-predictable process with finite variation, and the difference $\widetilde{X} = X - \Gamma(X)$ defines a $(\mathbb{Q}^\natural, \mathbb{G})$ local martingale.

We can check that the theorem of $s\mathcal{H}$-measure with covering condition is applicable in this (♮)-model. Actually let

$$\gamma_s = \frac{e^{-\Lambda_s}}{Z_s},\ \alpha_s = -\frac{e^{-\Lambda_s}}{1-Z_s},\ \beta_s = f(M_s^\tau - (1-Z_s)) + M_s^\tau f'(M_s^\tau - (1-Z_s)).$$

For $0 < a < \infty, n \in \mathbb{N}^*$, let

$$T_{a,n} = \inf\{v \geq a : \int_a^v (\gamma_w)^2 d\langle N\rangle_w > n, \text{ or } \int_a^v (\alpha_w)^2 d\langle N\rangle_w > n, \text{ or } \langle Y\rangle_v - \langle Y\rangle_a > n, \text{ or } v > a+n \}.$$

and the exponential martingale:

$$\eta^{a,n} = \mathcal{E}\left((-\gamma 1\!\!1_{[0,\tau]} - \alpha 1\!\!1_{(\tau,\infty)}) 1\!\!1_{(a,T_n]} \centerdot \widetilde{N} + (-\beta) 1\!\!1_{(\tau,\infty)} 1\!\!1_{(a,T_n]} \centerdot \widetilde{Y} \right).$$

We have that $\mathbb{Q}^\natural[\eta^{a,n}] = 1$ and the probability measure $\eta^{a,n} \cdot \mathbb{Q}^\natural$ is an $\mathcal{SH}$-measure on $(a, T_{a,n}]$. This can be verified by a direct computation using the above enlargement of filtration formula and Girsanov's theorem (the continuity of the martingales makes this computation straightforward).

Consider the intervals $(a, T_{a,n})$. Since $0 < Z < 1$ on $(0, \infty)$ and since N, Y are continuous, $\lim_{n\to\infty} T_{a,n} = \infty$. We have $(0, \infty) = \cup_{a\in\mathtt{Q}, n\in\mathbb{N}^*}(a, T_{a,n})$. The $\mathcal{SH}$-measure condition covering $[\tau, \infty)$ is satisfied. Consequently, the global optional splitting formula at the random time τ holds in this (♮)-model.

Multiple random times with marks

We now tackle the problem in its general form with multiple random times $\tau_1, \ldots, \tau_k$ with respectively marks $(\xi_1, \ldots, \xi_k)$. It is to note that, once the case of a single random time is well understood, the case of multiple random times can naturally be dealt with by induction. However, the multiplicity of random times may cause an inflation of notations in an induction argument. That is the true problem.

Multiple random times with marks

Let $m > 0$ be an integer and $\tau_1, \ldots, \tau_m$ be m random times. Let $(E, \mathcal{E})$ be a separable complete metric space with its Borel σ-algebra. Let $\triangle \in E$ and $E^\circ = E \setminus \{\triangle\}$. Let $(\xi_1, \ldots, \xi_m)$ be m random variables taking values in E°. Define, for $1 \leq i \leq m$,

$$H_i(t) = \begin{cases} \triangle & \text{if } t < \tau_i, \\ \xi_i & \text{if } \tau_i \leq t, \end{cases} \qquad t \geq 0,$$

and $\mathcal{H}_i(t) = \sigma(H_i(u) \colon 0 \leq u \leq t)$. Let $\overline{\mathcal{H}}_i(t) = \mathcal{N}^{\mathcal{H}_i(\infty)} \vee \mathcal{H}_i(t)$ and $\mathbb{H}_i = (\overline{\mathcal{H}}_i(t))_{t \geq 0}$.

Multiple random times with marks

Let $\mathcal{H}_t^{\{1,\ldots,m\}} = \sigma(H_i(s)\colon 1 \le i \le m,\ 0 \le s \le t)$ and

$$\mathcal{G}_t^{*m} = \mathcal{N}^{*m} \vee \cap_{s>t}(\mathcal{F}_s \vee \mathcal{H}_s^{\{1,\ldots,m\}}),$$

where $\mathcal{N}^{*m}$ denotes $\mathcal{N}^{\mathcal{H}_\infty^{\{1,\ldots,m\}} \vee \mathcal{F}_\infty}$. Let $\mathbb{G}^{*m}$ be the filtration of $\mathcal{G}_t^{*m}, t \ge 0$.
Let $\mathfrak{D}(E)$ be the space of all càdlàg functions taking values in E equipped with the Skorokhod topology and its Borel σ-algebra $\mathcal{D}$.

Multiple random times with marks

Notion. We say that the $\mathbb{G}^{*m}$-optional splitting formula holds at times $\tau_1, \ldots, \tau_m$ with respect to $\mathbb{F}$, if, for any $\mathbb{G}^{*m}$-optional process Y, there exist functions $Y^{(0)}, Y^{(1)}, \ldots, Y^{(m)}$ defined on $\mathfrak{D}(E)^m \times (\mathbb{R}_+ \times \Omega)$ being $\mathcal{D}^m \otimes \mathcal{O}(\mathbb{F})$-measurable such that

$$Y = \sum_{i=0}^{m} Y^{(i)}(H_1^{\sigma_{m,i}}, \ldots, H_m^{\sigma_{m,i}}) \mathbb{1}_{[\sigma_{m,i}, \sigma_{m,i+1})},$$

where $H_i^{\sigma_{m,i}}$ denotes the process H_i stopped at $\sigma_{m,i}$ and $(\sigma_{m,1}, \ldots, \sigma_{m,m})$ denotes the increasing re-ordering of $(\tau_1, \ldots, \tau_m)$.

Theorem. Suppose $m > 1$. Supose that $\mathbb{G}^{*m-1}$-optional splitting formula holds at times $\tau_1, \ldots, \tau_{m-1}$ with respect to $\mathbb{F}$. Suppose $\mathbb{G}^{*m}$-optional splitting formula holds at time τ_m with respect to $\mathbb{G}^{*m-1}$. Then, $\mathbb{G}^{*m}$-optional splitting formula holds at times $\tau_1, \ldots, \tau_m$ with respect to $\mathbb{F}$.

Notion. We say that $((\xi_1, \tau_1), \ldots, (\xi_m, \tau_m))$ satisfies the (strictly) positive (conditional) density hypothesis with respect to $\mathcal{F}_\infty$, if there exists a Borel probability measure ν on $E \times [0, \infty]$ and a stictly positive function γ^* on $(E \times [0, \infty])^m \times \Omega$ being $(\mathcal{E} \otimes \mathcal{B}[0, \infty])^m \otimes \mathcal{F}_\infty$ measurable such that

$$\mathbb{Q}[((\xi_1, \tau_1), \ldots, (\xi_m, \tau_m)) \in \mathtt{A} \,|\mathcal{F}_\infty] = \int_{\mathtt{A}} \gamma^*((x_1, t_1), \ldots, (x_m, t_m))\nu^{\otimes m}(\mathrm{d}(x$$

for any $\mathtt{A} \in (\mathcal{E} \times \mathcal{B}[0, \infty])^m$.

Proposition. If $m = 1$, if (ξ_1, τ_1) satisfies the density hypothesis with respect to $\mathcal{F}_\infty$, then the $\mathbb{G}^{*1}$-optional splitting formula holds at τ_1 with respect to $\mathbb{F}$.

Proof There exists a probability measure $\mathbb{Q}'$ equivalent to $\mathbb{Q}$, under which (ξ_1, τ_1) is independent of $\mathcal{F}_\infty$. We need only to prove the lemma under $\mathbb{Q}'$.

Let X be a bounded random variable in $\sigma(\xi_1, \tau_1)$. Let Y be a bounded random variable in $\mathcal{F}_\infty$. With independence, we can check that the process $\mathbb{Q}'[XY|\mathcal{G}^*_t]$, $t \geq 0$, satisfies the $\mathbb{G}^{*1}$-optional splitting formula at τ_1.

Now to complet the proof of the lemma, we only need to repeat the argument in [Dellecherie-Meyer Chapter XX, section 22]. ∎

Proposition. Suppose that $((\tau_1, \xi_1) \ldots, (\tau_m, \xi_m))$ satisfies the positive density hypothesis with respect to $\mathcal{F}_\infty$. Then, (τ_m, ξ_m) satisfies the positive density hypothesis with respect to $\mathcal{G}_\infty^{*(m-1)}$.
For any $1 \leq k < m$, $((\tau_1, \xi_1) \ldots, (\tau_k, \xi_k))$ satisfies the positive density hypothesis with respect to $\mathcal{F}_\infty$.
By the theorem of induction, we conclude.
Theorem. If the marked times $((\xi_1, \tau_1), \ldots, (\xi_m, \tau_m))$ satisfy the positive density hypothesis with respect to $\mathcal{F}_\infty$, then, $\mathbb{G}^{*m}$-optional splitting formula holds at times $\tau_1, \ldots, \tau_m$ with respect to $\mathbb{F}$.

Conclusion. The use in the literature of the optional splitting formula is justified, because it was always used under the density hypothesis.

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Thanks

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